Non-Relativistic Quantum Mechanics as a Gauge Theory

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Outline

State Functions

- In quantum mechanics, a particle is described by a complex-valued wave function, called state function $\psi : \mathbb{R}^{3+1} \longrightarrow \mathbb{C}^n$. The states ψ of a quantum mechanical system forms an infinite dimensional Hilbert space.
- The probability that a particle in a state ψ to be found inside the volume $V \subset \mathbb{R}^3$ is given by

$$\langle \psi | \psi \rangle = \int_V \psi^\dagger \psi d^3 x$$

where $\psi^\dagger = \bar{\psi}^t$.

- While the probability density $|\psi|^2$ is an observable, the state function $\psi(t, \mathbf{x})$ itself is not an observable. $\psi(\mathbf{r}, t)$ is a manifestation of a particle in a state.
- So, there is no physical reason why wave functions have to be $\mathbb{C}^n$ (complex vector)-valued functions!

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Holomorphic Tangent Bundle $T^+(\mathbb{C}^n)$

- We regard $\mathbb{C}^n$ as a Hermitian manifold of complex dimension n with the Hermitian metric

$$g = dz^\mu \otimes d\bar{z}^\mu .$$

- The complexified tangent bundle of $\mathbb{C}^n$, $T(\mathbb{C}^n)^{\mathbb{C}}$ is decomposed into holomorphic and anti-holomorphic tangent bundles of $\mathbb{C}$

$$T(\mathbb{C}^n)^{\mathbb{C}} = T^+(\mathbb{C}^n) \oplus T^-(\mathbb{C}^n).$$

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Lift of a Map

Definition

A map $h : X \longrightarrow Z$ is called a lift of $f : X \longrightarrow Y$ if there exists a map $g : Z \longrightarrow Y$ such that $f = g \circ h$.

$$\begin{array}{ccc} & & Z \\ & \nearrow & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Lifting a State Function

- Let $\phi : \mathbb{C}^n \longrightarrow T(\mathbb{C}^n)^{\mathbb{C}}$ be a vector field defined by

$$\phi(z^\mu, \bar{z}^\mu) = z^\mu \frac{\partial}{\partial z^\mu} + \bar{z}^\mu \frac{\partial}{\partial \bar{z}^\mu}.$$

- ϕ is a section of the complexified tangent bundle $T(\mathbb{C}^n)^{\mathbb{C}}$ since $\pi \circ \phi = Id$ where $\pi : T(\mathbb{C}^n)^{\mathbb{C}} \longrightarrow \mathbb{C}^n$ is the projection map.
- Given a state function $\psi : \mathbb{R}^{3+1} \longrightarrow \mathbb{C}^n$, define $\Psi : \mathbb{R}^{3+1} \longrightarrow T(\mathbb{C}^n)^{\mathbb{C}}$ by $\Psi = \phi \circ \psi$. Then Ψ is a lift of ψ since $\pi \circ \Psi = \psi$. We call Ψ a *lifted state function*.

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de Broglie Wave

Example

$\Psi(\mathbf{r},t) = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{\partial}{\partial z} + \bar{A}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{\partial}{\partial \bar{z}}$ is the lift of the de Broglie wave $\psi(\mathbf{r},t) = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, a plane wave that describes the motion of a free particle with momentum $\mathbf{p} = \mathbf{k}\hbar$.

The Probability of a Lifted State

- Recall that

$$g\left(\frac{\partial}{\partial z^\mu},\frac{\partial}{\partial z^\mu}\right)=g\left(\frac{\partial}{\partial \bar{z}^\mu},\frac{\partial}{\partial \bar{z}^\mu}\right)=0,\ g\left(\frac{\partial}{\partial z^\mu},\frac{\partial}{\partial \bar{z}^\mu}\right)=\frac{1}{2}.$$

- So, we obtain

$$|\psi|^2=g(\Psi,\Psi).$$

We define $g(\Psi,\Psi)$ to be the probability density of the lifeted state function Ψ.

- Since a state function and its lift have the same probability, we may study quantum mechanics with lifted state functions.

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Hermitian Structure h on the Holomorphic Tangent Bundle $T^+(\mathbb{C}^n)$

Hermitian structure always exists on the holomorphic tangent bundle $T^+(\mathbb{C}^n)$.

Theorem

For each $p \in \mathbb{C}^n$, define $h_p : T_p^+(\mathbb{C}^n) \times T_p^+(\mathbb{C}^n) \longrightarrow \mathbb{C}$ by

$$h_p(u,v) = g_p(u,\bar{v}) \text{ for } u,v \in T_p^+(\mathbb{C}^n).$$

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$T^+(\mathbb{C}^n)$ and Lifted States

- $\langle \Psi | \Psi \rangle = g(\Psi, \Psi) = g(\Psi^+, \overline{\Psi^+}) = h(\Psi^+, \Psi^+)$ where Ψ^+ is the holomorphic part $\Psi^+ = \psi^\mu \frac{\partial}{\partial z^\mu}$ of the lifted state Ψ.
- So without loss of generality, we may consider $\Psi^+ : \mathbb{R}^{3+1} \longrightarrow T^+(\mathbb{C}^n)$ as a lifted state function.

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Connection

- For an obvious reason, we would like to differentiate sections (fields). If we cannot differentiate them, we cannot do physics.
- Differentiating sections of a bundle can be done by introducing a *connection* ∇.
- In general, connection is not unique i.e. there is no unique way to differentiate sections and one needs to make a choice of connection.

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Hermitian Connection

Theorem

Let M be a Hermitian manifold. Given a holomorphic vector bundle $\pi : E \longrightarrow M$ and a Hermitian structure h, there exists a unique Hermitian connection.

Definition

A set of sections $\{e_1, \cdots e_n\}$ is called a unitary frame if

$$h(e_\mu, e_\nu) = \delta_{\mu\nu}.$$

Associated with a tangent bundle TM over a manifold M is a principal bundle called the frame bundle $LM = \bigcup_{p \in M} L_p M$, where $L_p M$ is the set of frames at $p \in M$. The structure group of the frame bundle LM is $U(n)$ or $SU(n)$ (if it is an oriented frame bundle).

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Hermitian Connection
Continued

Let $\{e_1, \cdots, e_n\}$ be a unitary frame. Define a local connection one-form $\omega = (\omega_\mu^\nu)$ by

$$\nabla e_\mu = \omega_\mu^\nu \otimes e_\nu.$$

Theorem

$\nabla^2 e_\mu = \nabla\nabla e_\mu = F_\mu^\nu e_\nu.$

F_μ^ν are the coefficients of the *curvature 2-form* F of the Hermitian connection ∇ or physically field strength which is defined by

$$F = d\omega + \frac{1}{2}\omega \wedge \omega.$$

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Let us differentiate ϕ^+ with the Hermitian connection ∇.

$$\begin{aligned}\nabla\phi^+ &= \nabla\left(z^\mu \frac{\partial}{\partial z^\mu}\right)\\ &= dz^\mu \otimes \frac{\partial}{\partial z^\mu} + z^\mu \nabla\left(\frac{\partial}{\partial z^\mu}\right)\\ &= dz^\mu \otimes \frac{\partial}{\partial z^\mu} + z^\mu \omega_\mu^\nu \otimes \frac{\partial}{\partial z^\nu}.\end{aligned}$$

This allows us to define a covariant derivative ∇^+ for lifted state functions Ψ^+.

Definition

$\nabla^+\Psi^+ = d\psi^\mu \otimes \frac{\partial}{\partial z^\mu} + \psi^\mu \omega_\mu^\nu \otimes \frac{\partial}{\partial z^\nu}.$

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Quantum Mechanics of a Charged Particle in an Electromagnetic Field

- Assume that $\omega \in \mathfrak{u}(1) = \mathfrak{so}(2)$. Then in terms of space-time coordinates (t, x^1, x^2, x^3), ω can be written as

$$\omega = -\frac{ie}{\hbar}\rho\, dt - \frac{ie}{\hbar}A_\alpha dx^\alpha,\ \ \alpha = 1, 2, 3.$$

- The covariant derivative $\nabla^+\psi^+$ of the lifted state function $\psi^+ : \mathbb{R}^{3+1} \longrightarrow \mathbb{C}$ is then

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- Let $\nabla_0 = \left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho\right)\frac{\partial}{\partial z}$, $\nabla_\alpha = \left(\frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha\right)\frac{\partial}{\partial z}$, $\alpha = 1, 2, 3$. Then the Schrödinger equation for a charge particle in an electromagnetic field is given by

$$i\hbar D_0\psi = -\frac{\hbar^2}{2m}D_\alpha^2\psi + V\psi,$$

where

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Questions?

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