Surfaces of Revolution with Constant Mean Curvature in Hyperbolic 3-Space

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Abstract

In this article, we construct surfaces of revolution with constant mean curvature $H = c$ in hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of constant curvature $-c^2$. It is shown that the limit of the surfaces of revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ is catenoid, the minimal surface of revolution in Euclidean 3-space as c approaches 0.

Introduction

Let $\mathbb{R}^3$ be equipped with the metric

$$ds^2 = (dt)^2 + e^{-2ct}\{(dx)^2 + (dy)^2\}. \qquad (1)$$

The space $(\mathbb{R}^3, g_c)$ has constant curvature $-c^2$. It is denoted by $\mathbb{H}^3(-c^2)$ and is called the *pseudospherical model* of hyperbolic 3-space. From the metric (1), one can easily see that $\mathbb{H}^3(-c^2)$ flattens out to $\mathbb{E}^3$, Euclidean 3-space as $c \to 0$.

In $\mathbb{H}^3(-c^2)$, surfaces of constant mean curvature $H = c$ are particularly interesting, because they exhibit many geometric properties in common with minimal surfaces in $\mathbb{E}^3$. This is not a coincidence. There is a one-to-one correspondence, so-called *Lawson correspondence*, between surfaces of constant mean curvature H_h in $\mathbb{H}^3(-c^2)$ and surfaces of constant mean curvature $H_e = \sqrt{H_h^2 - c^2}$ [Lawson]. Those corresponding constant mean curvature surfaces satisfy the same Gauss-Codazzi equations, so they share many geometric properties in common, even though they live in different spaces. In this article, we are particularly interested in constructing surfaces of revolution with $H = c$ in $\mathbb{H}^3(-c^2)$. Hyperbolic 3-space does not have rotational symmetry as much as Euclidean 3-space does. From the metric (1), we see that rotations on the xy-plane i.e. rotations about the t-axis may be considered in $\mathbb{H}^3(-c^2)$. Surfaces of constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ can be in general constructed by Bryant's representation formula which is an analogue of Weierstrass representation formula for minimal surfaces in $\mathbb{E}^3$[Bryant]. But it is not suitable to use to construct surfaces of revolution with $H = c$. We calculate directly the mean curvature H of the surface obtained by rotating an unknown profile curve about the $t-axis$. This results a second order non-linear differential equation of the profile curve. We unfortunately cannot solve the differential equation analytically but are able to solve it numerically with the aid of MAPLE software. Once we obtain the profile curve, we then construct surface of revolution with $H = c$ simply by rotating the profile curve about the t-axis. From the differential equation of profile curves, it can be seen that the limit of the surfaces of revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ is a catenoid, the minimal surface of revolution in Euclidean 3-space as c approaches 0. This limiting behavior of the surfaces of revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ is also illustrated with graphics.

The author is a junior mathematics and physics major of the University of Southern Mississippi. This research has been conducted under the direction of Dr. Sungwook Lee in the Department of Mathematics at the University of Southern Mississippi.

1 Parametric Surfaces in $\mathbb{H}^3(-c^2)$

Let M be a domain and $\varphi : M \to \mathbb{H}^3(-c^2)$ a parametric surface. The metric (1) induces an inner product on each tangent space $T_p\mathbb{H}^3(-c^2)$. This inner product can be used to define conformal surfaces in $\mathbb{H}^3(-c^2)$.

Definition 1. $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ is said to be conformal if

$$\langle \varphi_u, \varphi_v \rangle = 0, \quad |\varphi_u| = |\varphi_v| = e^{\omega/2}, \qquad (2)$$

where (u, v) is a local coordinate system in M and $\omega : M \to \mathbb{R}$ is a real-valued function in M. The induced

metric on the conformal parametric surface is given by

$$ds_\varphi^2 = e^\omega\{(du)^2 + (dv)^2\}. \tag{3}$$

In order to calculate the mean curvature of φ, we need to find a unit normal vector field N of φ. For that, we need something like cross product. $\mathbb{H}^3(-c^2)$ is not a vector space but we can define an analogue[1] of cross product locally on each tangent space $T_p\mathbb{H}^3(-c^2)$. Let $\mathbf{v} = v_1\left(\frac{\partial}{\partial t}\right)_p + v_2\left(\frac{\partial}{\partial x}\right)_p + v_3\left(\frac{\partial}{\partial y}\right)_p$, $\mathbf{w} = w_1\left(\frac{\partial}{\partial t}\right)_p + w_2\left(\frac{\partial}{\partial x}\right)_p + w_3\left(\frac{\partial}{\partial y}\right)_p \in T_p\mathbb{H}^3(-c^2)$, where $\left\{\left(\frac{\partial}{\partial t}\right)_p, \left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p\right\}$ denote the canonical basis for $T_p\mathbb{H}^3(-c^2)$. The cross product $\mathbf{v} \times \mathbf{w}$ is then defined to be

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\left(\frac{\partial}{\partial t}\right)_p$$
$$+ e^{2ct}(v_3 w_1 - v_1 w_3)\left(\frac{\partial}{\partial x}\right)_p \tag{4}$$
$$+ e^{2ct}(v_1 w_2 - v_2 w_1)\left(\frac{\partial}{\partial y}\right)_p$$

where $p = (t, x, y) \in \mathbb{H}^3(-c^2)$.

Let

$$E := \langle \varphi_u, \varphi_u \rangle, \ \ F := \langle \varphi_u, \varphi_v \rangle, \ \ G := \langle \varphi_v, \varphi_v \rangle.$$

Then by a direct calculation we obatin

Proposition 2. *Let $\varphi : M \to \mathbb{H}^3(-c^2)$ be a parametric surface. Then on each tangent plane $T_p\varphi(M)$, we have*

$$||\varphi_u \times \varphi_v||^2 = e^{4ct(u,v)}(EG - F^2) \tag{5}$$

where $p = (t(u,v), x(u,v), y(u,v)) \in \mathbb{H}^3(-c^2)$.

Remark 3. If $c \to 0$, (5) becomes the familiar formula

$$||\varphi_u \times \varphi_v||^2 = EG - F^2$$

from the Euclidean case.

2   The Mean curvature of a Parametric Surface in $\mathbb{H}^3(-c^2)$

In the Euclidean case, the mean curvature of a parametric surface $\varphi(u,v)$ may be calculated by Gauss' formula

$$H = \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)} \tag{6}$$

[1]We will simply call it cross prodcut.

where

$$\ell = \langle \varphi_{uu}, N \rangle, \mathfrak{m} = \langle \varphi_{uv}, N \rangle, \mathfrak{n} = \langle \varphi_{vv}, N \rangle$$

and N is a unit normal vector field of φ. It is not certain, but (6) does not appear to be valid for parametric surfaces in $\mathbb{H}^3(-c^2)$ in general. The derivation of (6) requires the use of Lagrange's identity, but it is no longer valid in the tangent spaces of $\mathbb{H}^3(-c^2)$. However, (6) is still valid for conformal surfaces in $\mathbb{H}^3(-c^2)$. It is well-known that:

Proposition 4. *Let $\varphi : M \to \mathbb{H}^3(-c^2)$ be a conformal surface satisfying (2). The mean curvature H of φ is then computed to be*

$$H = \frac{1}{2}e^{-\omega}\langle \Delta\varphi, N \rangle. \tag{7}$$

One can then easily see that the the formulas (6) and (7) coincide for conformal surfaces.

3   Surfaces of Revolution with Constant Mean Curvature $H = c$ in $\mathbb{H}^3(-c^2)$

In this section, we construct a surface of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$. As mentioned in Introduction, rotations about the t-axis are the only type of Euclidean rotations that can be considered in $\mathbb{H}^3(-c^2)$.

Consider a profile curve $\alpha(u) = (u, h(u), 0)$ in the tx-plane. Denote $\varphi(u, v)$ as the rotation of $\alpha(u)$ about the t-axis through an angle v. Then,

$$\varphi(u, v) = (u, h(u)\cos v, h(u)\sin v). \tag{8}$$

The quantities E, F, and G are calculated to be

$$\begin{aligned} E &= e^{-2cu}\{e^{2cu} + (h'(u))^2\}, \\ F &= 0, \\ G &= e^{-2cu}h(u). \end{aligned}$$

If we require $\varphi(u, v)$ to be conformal, then

$$e^{2cu} + (h'(u))^2 = (h(u))^2. \tag{9}$$

The quantities $\ell, \mathfrak{m}, \mathfrak{n}$ are calculated to be

$$\begin{aligned} \ell &= -\frac{h"(u)h(u)}{\sqrt{(h(u))^2(e^{2cu} + (h'(u))^2)}}, \\ \mathfrak{m} &= 0, \\ \mathfrak{n} &= \frac{(h(u))^2}{\sqrt{(h(u))^2(e^{2cu} + (h'(u))^2)}}. \end{aligned}$$

The mean curvature H is computed to be[2].

$$H = \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)}$$
$$= \frac{1}{2} \frac{-h(u)h''(u) + e^{2cu} + (h'(u))^2}{e^{-2cu}(h(u))^3}.$$

If we apply the conformality condition (**??**), H becomes

$$H = \frac{-h''(u) + h(u)}{2e^{-2cu}(h(u))^3}. \tag{10}$$

Let $H = c$. Then (10) can be written as

$$h''(u) - h(u) + 2ce^{-2cu}(h(u))^3 = 0. \tag{11}$$

Hence, constructing a surface of revolution with $H = c$ comes down to solving the second order nonlinear differential equation (11). If $c \to 0$, then (11) becomes

$$h''(u) - h(u) = 0 \tag{12}$$

which is a harmonic oscillator. This is the profile curve for a surface of revolution in $\mathbb{E}^3$. (12) has the general solution

$$h(u) = c_1 \cosh u + c_2 \sinh u.$$

For $c_1 = 1, c_2 = 0$, $\varphi(u, v)$ is given by

$$\varphi(u, v) = (u, \cosh u \cos v, \cosh u \sin v) \tag{13}$$

This is a minimal surface of revolution in $\mathbb{E}^3$, which is called a *catenoid* since it is obtained by rotating a catenary $h(u) = \cosh u$. See Figure 1.

Unfortunately, the author cannot solve (11) analytically, so we solve it numerically with the aid of MAPLE.

4 The Illustration of the Limit of Surfaces of Revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ as $c \to 0$

In section 3, it is shown that the limit of surfaces of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ is a catenoid, a minimal surface of revolution in $\mathbb{E}^3$. Such limiting behavior of surfaces of revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ is illustrated with graphics in Figure 2 ($H = 1$), Figure 3 ($H = \frac{1}{2}$), Figure 4 ($H = \frac{1}{4}$), Figure 5 ($H = \frac{1}{8}$), Figure 6 ($H = \frac{1}{64}$), Figure 7 ($H = \frac{1}{256}$). Figure 7 (b) already looks pretty close to the catenoid in Figure 1.

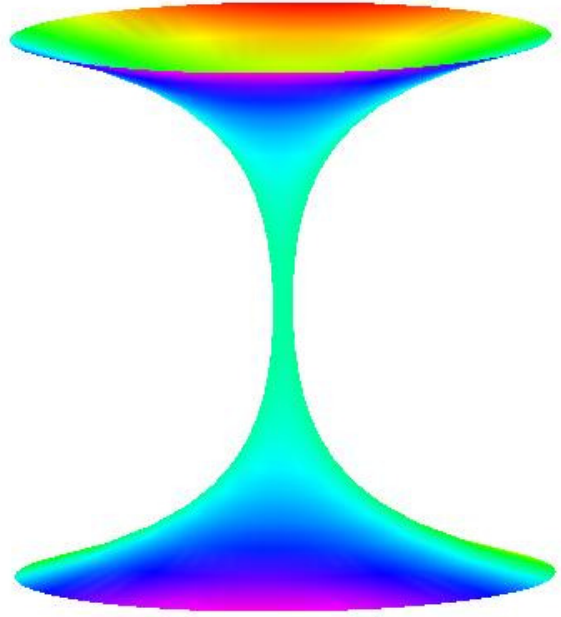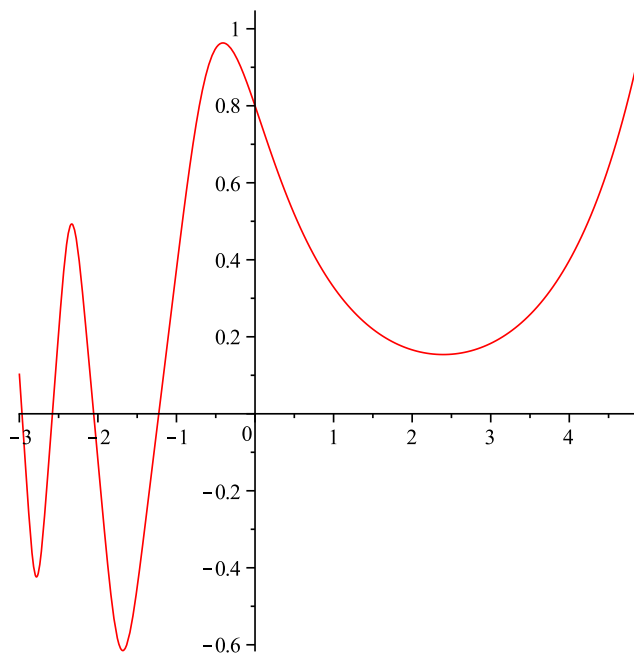[2]The validity of this formula should not be a concern since we assume that the surface is conformal.
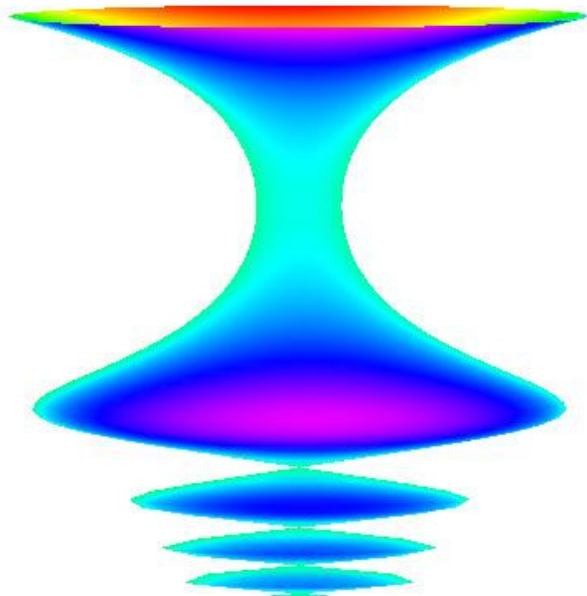

Figure 1: Catenoid in $\mathbb{E}^3$

References

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[Bryant] Robert L. Bryant, *Surfaces of mean curvature one in hyperbolic 3-space*, Astérisque **12**, No. 154-155, 321-347 (1988)
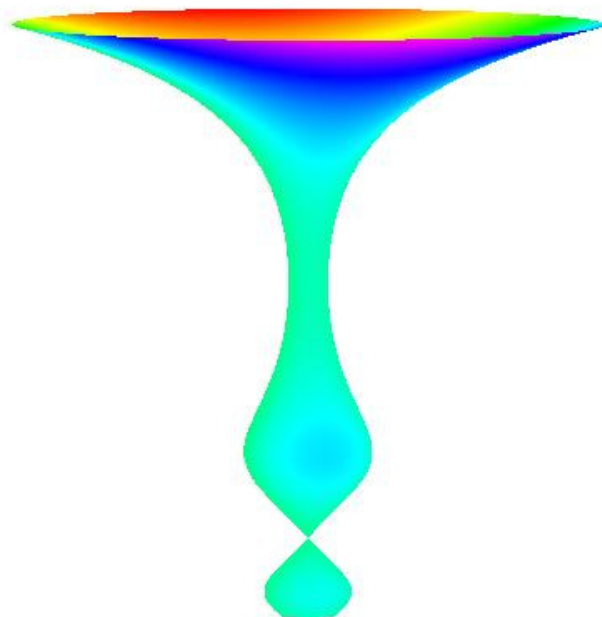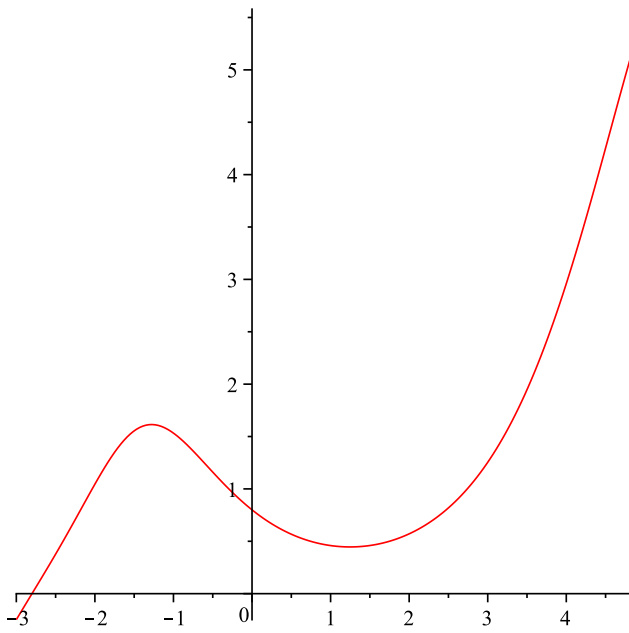
(a) Profile curve $h(u)$

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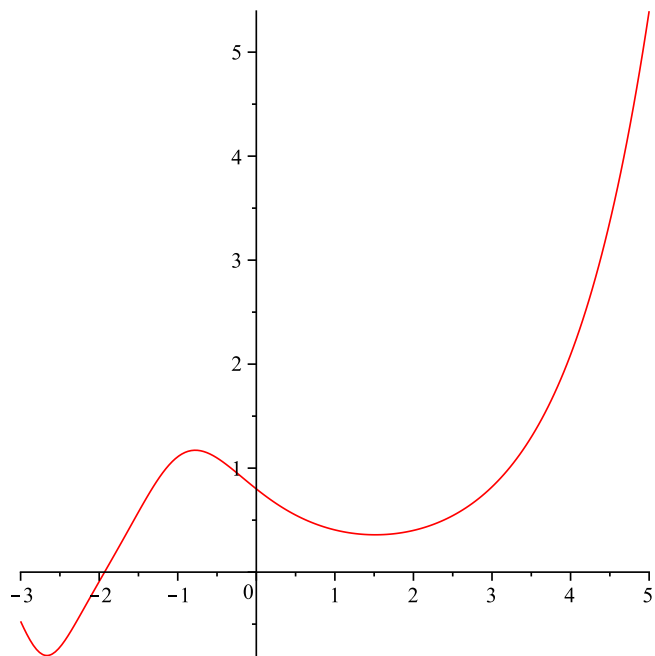
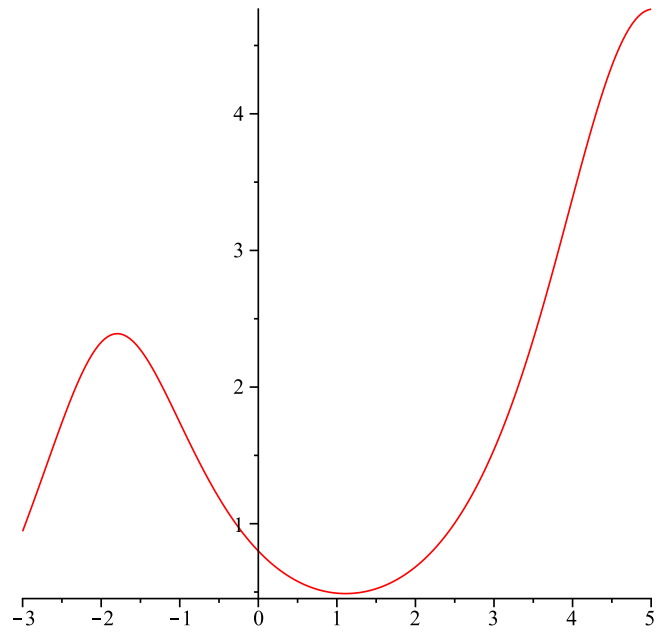(b) Surface of Revolution in $\mathbb{H}^3(-c^2)$

(b) Surface of Revolution in $\mathbb{H}^3(-c^2)$

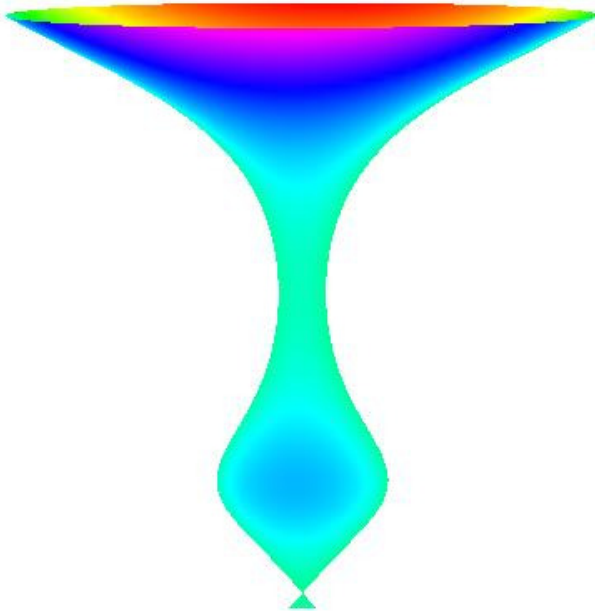Figure 2: Constant Mean Curvature $H = 1$

Figure 3: Constant Mean Curvature $H = \frac{1}{2}$

(a) Profile Curve $h(u)$

(a) Profile Curve $h(u)$

(b) Surface of Revolution in $\mathbb{H}^3(-c^2)$

(b) Surface of Revolution in $\mathbb{H}^3(-c^2)$

Figure 4: Constant Mean Curvature $H = \frac{1}{4}$
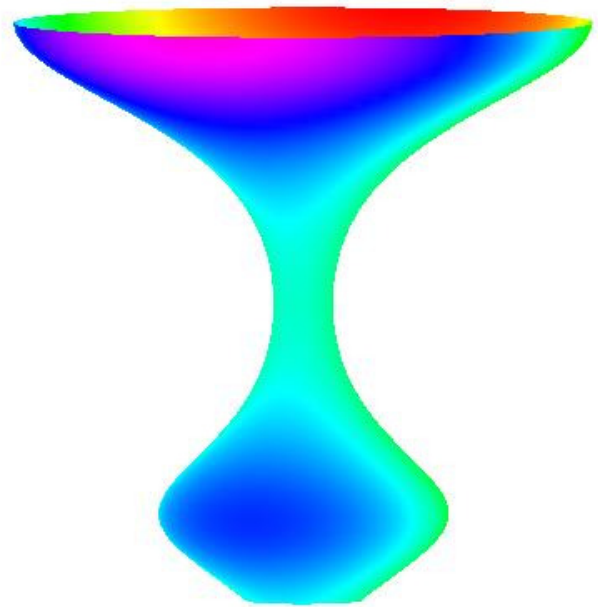
Figure 5: Constant Mean Curvature $H = \frac{1}{8}$
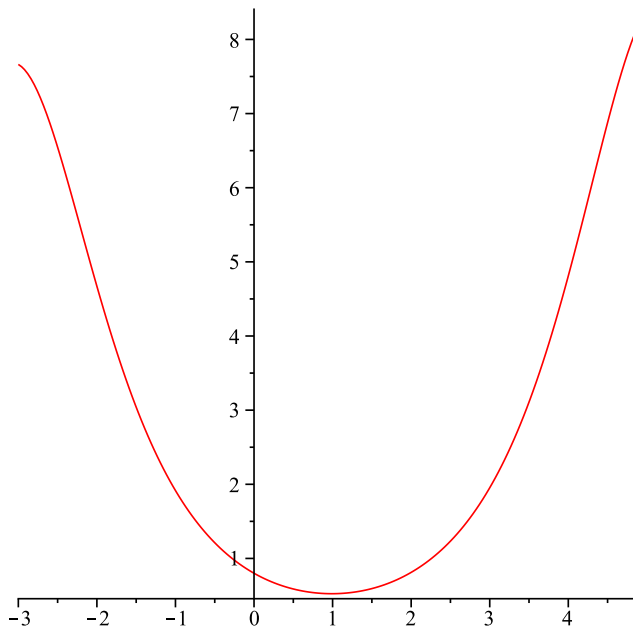
(a) Profile Curve $h(u)$

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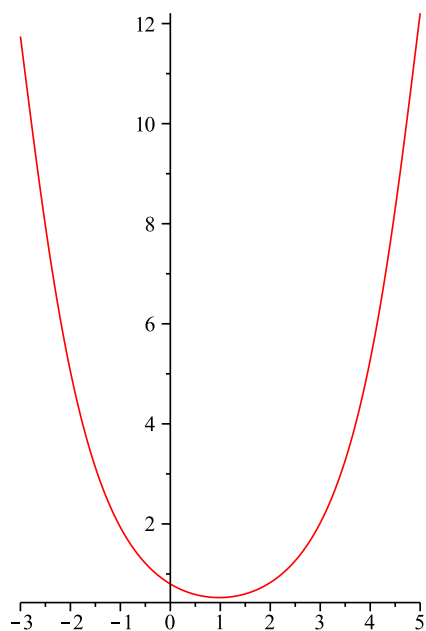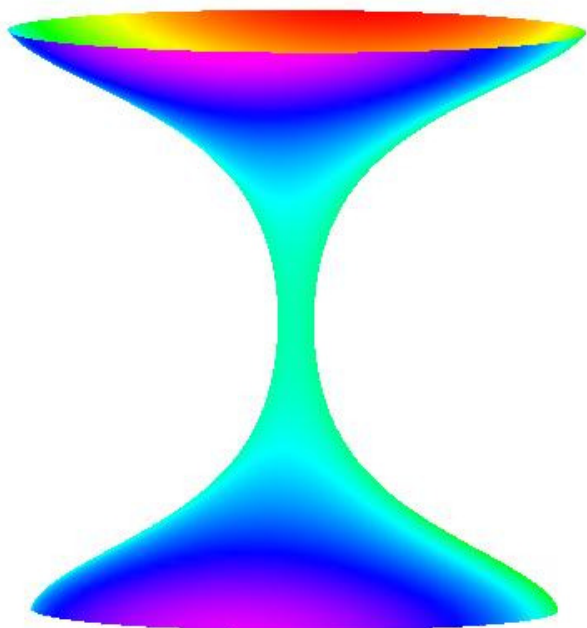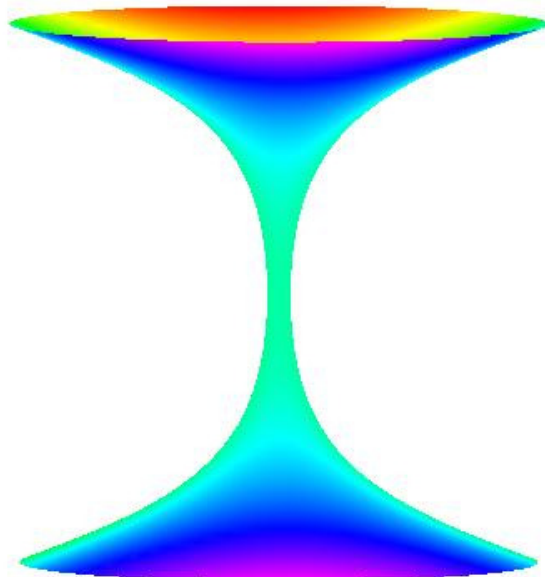(b) Surface of Revolution in $\mathbb{H}^3(-c^2)$

Figure 6: ConstSurface of Revolution in $\mathbb{H}^3(-c^2)$ant Mean Curvature $H = \frac{1}{64}$

(b) Surface of Revolution in $\mathbb{H}^3(-c^2)$

Figure 7: Constant Mean Curvature $H = \frac{1}{256}$