Surfaces of Revolution in Hyperbolic 3-Space

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Outline

Hyperbolic 3-Space $\mathbb{H}^3(-c^2)$

- Let $\mathbb{R}^3$ be equipped with the metric

$$g_c = (dt)^2 + e^{-2ct}\left\{(dx)^2 + (dy)^2\right\}$$

where c is a constant.

- $(\mathbb{R}^3, g_c)$ has constant curvature $-c^2$ and is denoted by $\mathbb{H}^3(-c^2)$.
- $\mathbb{H}^3(-c^2)$ is called the *pseudospherical model* of hyperbolic 3-space.
- As $c \to 0$, $\mathbb{H}^3(-c^2)$ flattens out to $\mathbb{E}^3$, Euclidean 3-space.

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Conformal Parametric Surfaces in $\mathbb{H}^3(-c^2)$

Definition

A parametric surface $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ is said to be *conformal* if

$$\langle \varphi_u, \varphi_v \rangle = 0, |\varphi_u| = |\varphi_v| = e^{\omega/2},$$

where (u, v) is a local coordinate system in M and $\omega : M \to \mathbb{R}$ is a real-valued function in M.

The induced metric on the conformal parametric surface is given by

$$ds_\varphi^2 = e^\omega \left\{ (du)^2 + (dv)^2 \right\}.$$

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Cross Product in $T_p\mathbb{H}^3(-c^3)$

$\mathbb{H}^3(-c^2)$ is not a vector space but each tangent space $T_p\mathbb{H}^3(-c^2)$ is, and we can consider cross product on each $T_p\mathbb{H}^3(-c^2)$.

Let $\mathbf{v} = v_1 \left(\frac{\partial}{\partial t}\right)_p + v_2 \left(\frac{\partial}{\partial x}\right)_p + v_3 \left(\frac{\partial}{\partial y}\right)_p$,
$\mathbf{w} = w_1 \left(\frac{\partial}{\partial t}\right)_p + w_2 \left(\frac{\partial}{\partial x}\right)_p + w_3 \left(\frac{\partial}{\partial y}\right)_p \in T_p\mathbb{H}^3(-c^2)$, where
$\left\{\left(\frac{\partial}{\partial t}\right)_p, \left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p\right\}$ denote the canonical basis for $T_p\mathbb{H}^3(-c^2)$. Then:

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Cross Product in $T_p\mathbb{H}^3(-c^3)$

Continued

Definition

The cross product $\mathbf{v} \times \mathbf{w}$ is defined by

$$\begin{aligned}\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2) \left(\frac{\partial}{\partial t}\right)_p \\ + e^{2ct}(v_3 w_1 - v_1 w_3) \left(\frac{\partial}{\partial x}\right)_p \\ + e^{2ct}(v_1 w_2 - v_2 w_1) \left(\frac{\partial}{\partial y}\right)_p,\end{aligned}$$

where $p = (t, x, y) \in \mathbb{H}^3(-c^2)$.

The Mean Curvature of a Conformal Parametric Surface in $\mathbb{H}^3(-c^2)$

If a parametric surface $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ is conformal, the mean curvature H is computed by the formula

$$H = \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)},$$

where

$$E = \langle \varphi_u, \varphi_u \rangle, \ F = \langle \varphi_u, \varphi_v \rangle, \ G = \langle \varphi_v, \varphi_v \rangle$$
$$\ell = \langle \varphi_{uu}, N \rangle, \ \mathfrak{m} = \langle \varphi_{uv}, N \rangle, \ \mathfrak{n} = \langle \varphi_{vv}, N \rangle$$

and $N = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}$ is a unit normal vector field on φ.

Rotations in $\mathbb{H}^3(-c^2)$

- Rotations about the t-axis are the only type of Euclidean rotations that can be considered in $\mathbb{H}^3(-c^2)$.
- The rotation of a profile curve $\alpha(u) = (u, h(u), 0)$ in the tx-plane about the t-axis through an angle v:

$$\varphi(u,v) = (u, h(u)\cos v, h(u)\sin v).$$

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Differential Equation of $h(u)$ for Surfaces of Revolution with CMC $H=c$ in $\mathbb{H}^3(-c^2)$

- The mean curvature H of a conformal surface of revolution in $\mathbb{H}^3(-c^2)$ is computed to be

$$H = \frac{-h''(u)+h(u)}{2e^{-2cu}(h(u))^3}.$$

- By setting $H=c$, we obtain the second order non-linear differential equation of $h(u)$

$$h''(u)-h(u)+2ce^{-2cu}(h(u))^3=0.$$

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Lawson Correspondence

- There is a one-to-one correspondence, so-called *Lawson correspondence*, between surfaces of constant mean curvature H_h in $\mathbb{H}^3(-c^2)$ and surfaces of constant mean curvature $H_e = \sqrt{H_h^2 - c^2}$ in $\mathbb{E}^3$. [H. Blaine Lawson, Jr., *Complete minimal surfaces in S^3*, Ann. of Math. **92**, 335-374 (1970)]
- In particular, there is a one-to-one correspondence between surfaces of constant mean curvature $H = c$ surfaces in $\mathbb{H}^3(-c^2)$ and minimal surfaces in $\mathbb{E}^3$.

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Limit Behavior of Surfaces of Revolution with CMC $H = c$ as $c \to 0$

- If $c \to 0$, then the differential equation of $h(u)$ becomes

$$h''(u) - h(u) = 0,$$

which is a harmonic oscillator. Its solution is

$$h(u) = c_1 \cosh u + c_2 \sinh u.$$

- For $c_1 = 1$, $c_2 = 0$, we obtain the catenoid

$$\varphi(u, v) = (u, \cosh u \cos v, \cosh u \sin v),$$

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Catenoid in $\mathbb{E}^3$

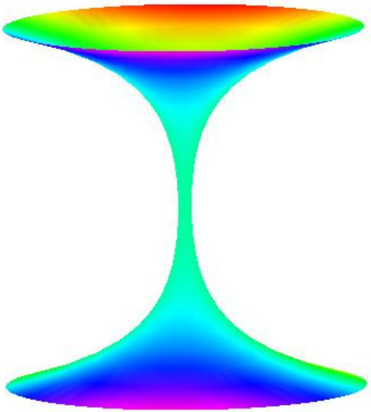

Figure: Catenoid in $\mathbb{E}^3$

Surface of Revolution with CMC $H = 1$ in $\mathbb{H}^3(-1)$

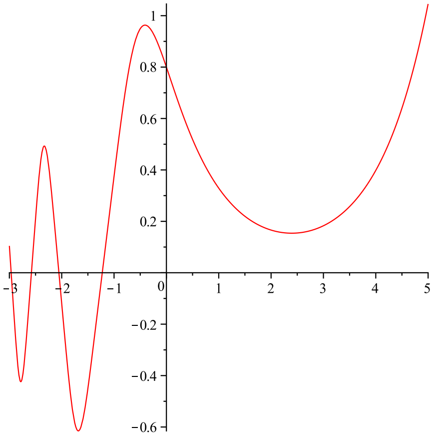

Figure: CMC $H = 1$: Profile Curve

Surface of Revolution with CMC $H=1$ in $\mathbb{H}^3(-1)$

Continued

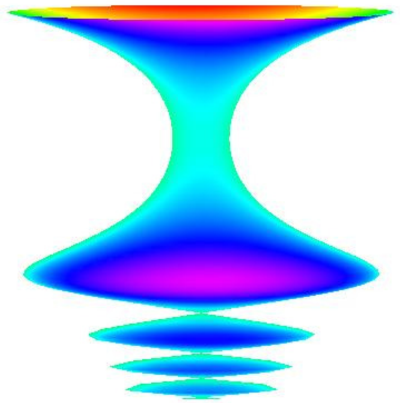

Figure: CMC $H=1$: Surface of Revolution

Surface of Revolution with CMC $H = \frac{1}{4}$ in $\mathbb{H}^3\left(-\frac{1}{16}\right)$

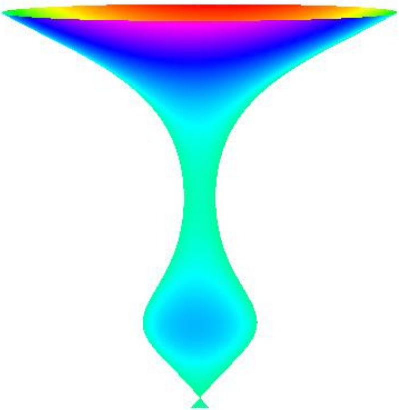

Figure: CMC $H = \frac{1}{4}$: Surface of Revolution

Surface of Revolution with CMC $H=\frac{1}{8}$ in $\mathbb{H}^3\left(-\frac{1}{64}\right)$

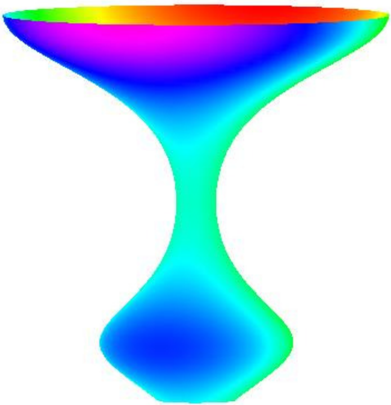

Figure: CMC $H=\frac{1}{8}$: Surface of Revolution

Surface of Revolution with CMC $H = \frac{1}{256}$ in $\mathbb{H}^3\left(-\frac{1}{65536}\right)$

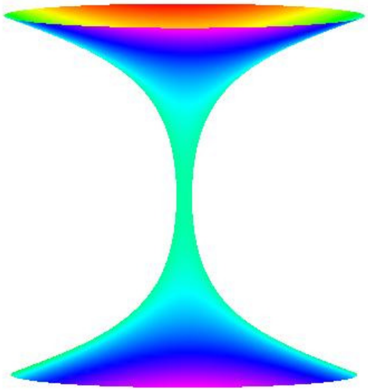

Figure: CMC $H = \frac{1}{256}$: Surface of Revolution

Animations

- Animation of Profile Curves $h(u)$
  http://www.math.usm.edu/lee/profileanim.gif
- Animation of Surfaces of Revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$
  http://www.math.usm.edu/lee/cmcanim.gif
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Questions?

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