ON A STOCHASTIC MODEL OF EPIDEMICS

by

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ABSTRACT

This thesis examines a stochastic model of epidemics initially proposed and studied by Norman T. J. Bailey [1]. We discuss some issues with Bailey's stochastic model and argue that it may not be a viable theoretical platform for a more general epidemic model. A possible alternative approach to the solution of Bailey's stochastic model and stochastic modeling is proposed as well. Regrettably, any further study on those proposals will have to be discussed elsewhere due to a time constraint.

ACKNOWLEDGMENTS

This is a thank you to my family because I would not have made it this far without their support. I am forever indebted to my advisor, who was vital in helping me stay on track while helping me build my confidence throughout this study.

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Chapter 1

Some Preliminaries

In order to make this thesis self-contained, we include in this chapter some preliminary materials that may be referred to or needed in our later discussions.

1.1 The Poisson Process

The *Poisson process* is a stochastic process in which a specific event happens at a fixed rate but entirely at random. Let us consider some population impacted by an epidemic at time t with a discrete random variable $X(t)$, i.e., let $X(t)$ be the total number of individuals in the population at time t. The probability that the number of the infected population is n at time t is denoted by

$$P\{X(t) = n\} = p_n(t)$$

for $n = 0, 1, 2, \cdots$. We suppose no one in the population is infected at time $t = 0$, so $p_0(0) = 1$. Next, we suppose that the chance of a new infection occurring in any short time interval is independent of the previous state of the population and the present state. This new chance of infection in Δt can be written as $\lambda \Delta t + \sigma(\Delta t)$. Here we have that λ is a constant and $\sigma(\Delta t)$ is the chance of two or more simultaneous infections. The chance of no change in the population would be $1 - \Delta t - \sigma(\Delta t)$. Therefore, we obtain,

$$p_n(t + \Delta t) = p_{n-1}(t)\lambda \Delta t + p_n(t)(1 - \lambda \Delta t) \tag{1.1}$$

where the terms that are small compared to Δt have been disregarded. Now,

$$\frac{dp_n(t)}{dt} = \lim_{\Delta t \to 0} \frac{p_n(t + \Delta t) - p_n(t)}{\Delta t}. \tag{1.2}$$

Using (1.1), (1.2) is simplified to the differential-difference equation

$$\frac{dp_n(t)}{dt} = \lambda [p_{n-1}(t) - p_n(t)] \tag{1.3}$$

where $n > 0$. Note that we have $n = 0$ at $t + \Delta t$ only if $n = 0$ at time t and no new infections have occurred in Δt. This implies that we have

$$p_0(t + \Delta t) = p_0(t)(1 - \lambda \Delta t), \tag{1.4}$$

which gives us the differential equation

$$\frac{dp_0}{dt} = -\lambda p_0(t). \tag{1.5}$$

Now, we can solve (1.5) simply by integration to obtain

$$p_0(t) = e^{-\lambda t} \tag{1.6}$$

with the initial condition $p_0(0) = 1$. Starting with $p_0(t)$, by solving (1.3) iteratively for $n = 1, 2, \cdots$, we obtain

$$p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, n = 0, 1, 2, \cdots \tag{1.7}$$

which is a *Poisson distribution* with the parameter λt.

While getting successive solutions of the Poisson distribution is easy and simple, it is not guaranteed that it will always work in general. There is, however, another method of obtaining probability distributions. Let $P(x,t) = \sum_{n=0}^{\infty} p_n(t)x^n$. Then $P(x,t)$ is called a *probability generating function* for the probability distribution $p_n(t)$. The equations (1.3) and (1.4) can be combined to

$$\frac{dp_n(t)}{dt} = \lambda\{p_{n-1}(t) - p_n(t)\},\ n \geq 0 \tag{1.8}$$

with $p_{-1}(t) := 0$ and $p_0(0) = 1$. Multiplying (1.8) by x^n and summing it over n from $n = 1$ to ∞, we obtain the differential equation

$$\frac{\partial P(x,t)}{\partial t} = \lambda(x-1)P(x,t) \tag{1.9}$$

with $P(x,0) = 1$. (1.9) is a simple separable equation, and its solution is given by

$$\begin{aligned} P(x,t) &= e^{\lambda t(x-1)} \\ &= e^{\lambda tx} e^{-\lambda t} \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda tx)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} x^n \end{aligned}$$

From this, we retrieve the Poisson distribution (1.7).

1.2 Hypergeometric Functions

The second-order linear differential equation

$$x(x-1)y''(x)+[(a+b+1)x-c]y'(x)+aby(x)=0 \tag{1.10}$$

is called the *hypergeometric equation*. The hypergeomtric equation has regular singularities at $x=0,1,\infty$. A solution to (1.10) is given by

$$\begin{aligned} y(x) &= {}_2F_1(a,b,c;x) \\ &= 1+\frac{ab}{c}\frac{x}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!}+\cdots \end{aligned} \tag{1.11}$$

where $c\neq 0,-1,-2,-3,\cdots$. The solution ${}_2F_1(a,b,c;x)$ is called the *hypergeometric function* or the *hypergeometric series*. The range of convergence is $|x|<1$ and $x=1$, for $c>a+b$, and $x=-1$, for $c>a+b-1$. Using *Pochhammer symbols*

$$\begin{aligned} &(a)_n = a(a+1)(a+2)\cdots(a+n-1)=\frac{(a+n-1)!}{(a-1)!} \\ &(a)_0 = 1 \end{aligned} \tag{1.12}$$

we can write the hypergeometric function as

$${}_2F_1(a,b,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \tag{1.13}$$

Proposition 1.2.1.

$$\frac{d}{dx}[{}_2F_1(a,b,c;x)]=\frac{ab}{c}{}_2F_1(a+1,b+1,c+1;x) \tag{1.14}$$

Proof.

$$\begin{aligned} \frac{d}{dx}[{}_2F_1(a,b,c;x)] &= \frac{d}{dx}\left[1+\frac{ab}{c}\frac{x}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{x^3}{3!}+\cdots\right] \\ &= \frac{ab}{c}\left[1+\frac{a(a+1)(b+1)}{(c+1)}\frac{x}{1!}+\frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)}\frac{x^2}{2!}+\cdots\right] \\ &= \frac{ab}{c}\frac{ab}{c}{}_2F_1(a+1,b+1,c+1;x) \end{aligned}$$

More details and other properties on hypergeometric functions can be found in [4] and Volume II of [3].

Chapter 2

Main Discussions

2.1 A Deterministic Model of Epidemics

In this section, we introduce a deterministic model of epidemics. The assumptions we will use for this model are that a population comprises of homogeneously mixed individuals. Also, the size of this population will be $n+1$. An epidemic starts when the first individual becomes infectious, which happens when $t = 0$, where t is time. Now, let x represent the number of susceptibles, individuals able to get sick, and y is the number of infectives. The equation representing the entire populations is now $x+y = n+1$. Finally, the rate of infection is proportional to the number of infectives and susceptibles. These assumptions lead us to the differential equation

$$\frac{dx}{dt} = -\beta xy = -\beta x(n-x+1) \tag{2.1}$$

where β is the constant of proportionality called the *infection rate*. For the sake of simplicity, we introduce a new time variable $\tau = \beta t$. The equation (2.1) can be written as

$$\frac{dx}{d\tau} = -x(n-x+1). \tag{2.2}$$

According to the assumption, the initial condition is given by $x(0) = n$. With this initial condition, the solution $x(\tau)$, the number of susceptibles at time τ, is obtained as follows.

$$x(\tau) = \frac{n(n+1)}{n+e^{(n+1)\tau}}. \tag{2.3}$$

From $x+y = n+1$, we also find the number of infectives $y(\tau)$ at time τ

$$y(\tau) = \frac{n+1}{1+ne^{-(n+1)\tau}}. \tag{2.4}$$

This model can be easily generalized to the case where an epidemic starts with a infectives. In this case, x and y satisfy the relationship $x+y = n+a$. Accordingly, the equation (2.1) is now modified to

$$\frac{dx}{dt} = -\beta xy = -\beta x(n-x+a) \tag{2.5}$$

and with the time variable $\tau = \beta t$, (2.5) is simplified to

$$\frac{dx}{d\tau} = -x(n-x+a). \tag{2.6}$$

The solution $x(\tau)$, the number of susceptibles at τ, to (2.6) is

$$x(\tau) = \frac{n(n+a)}{n+ae^{(n+a)\tau}} \tag{2.7}$$

and the number of infectives $y(\tau)$ is

$$y(\tau) = \frac{a(n+a)}{a+ne^{-(n+a)\tau}}. \tag{2.8}$$

In a mathematical model of epidemics, the *epidemic curve* $\omega(\tau)$, the rate at which new infections occur, is a significant quantity. In particular, the epidemic curve may allow us to predict when the rate will reach the peak. By definition, $\omega(\tau) = \frac{dy}{d\tau}$. Using the relationship $x+y = n+1$, it can be written as

$$\omega(\tau) = -\frac{dx}{d\tau} = xy. \tag{2.9}$$

From equations (2.3) and (2.4), the epidemic curve is given by

$$\omega(\tau) = \frac{n(n+1)^2e^{(n+1)\tau}}{(n+e^{(n+1)\tau})^2}. \tag{2.10}$$

Figure 2.1 shows two epidemic curves with $n = 20$ (red) and for $n = 30$ (blue) for $0 \le t \le 1$.

$\frac{d\omega}{d\tau} = xy(y-x)$, so $\frac{d\omega}{d\tau} = 0$ implies that $x = y$ from which we determine $\tau = \frac{\ln n}{n+1}$. That is, the epidemic curve $\omega(\tau)$ peaks at $\tau = \frac{\ln n}{n+1}$ and the maximum rate of infection is $\omega(\frac{\ln n}{n+1}) = \frac{1}{4}(n+1)^2$.

For the generalized case (2.6), the corresponding epidemic curve is given by

$$\omega(\tau) = \frac{an(n+a)^2e^{(n+a)\tau}}{[n+ae^{(n+a)\tau}]^2}. \tag{2.11}$$

2.2 A Simple Stochastic Model

This section examines a simple stochastic model initially introduced by Norman T. J. Bailey in [1].

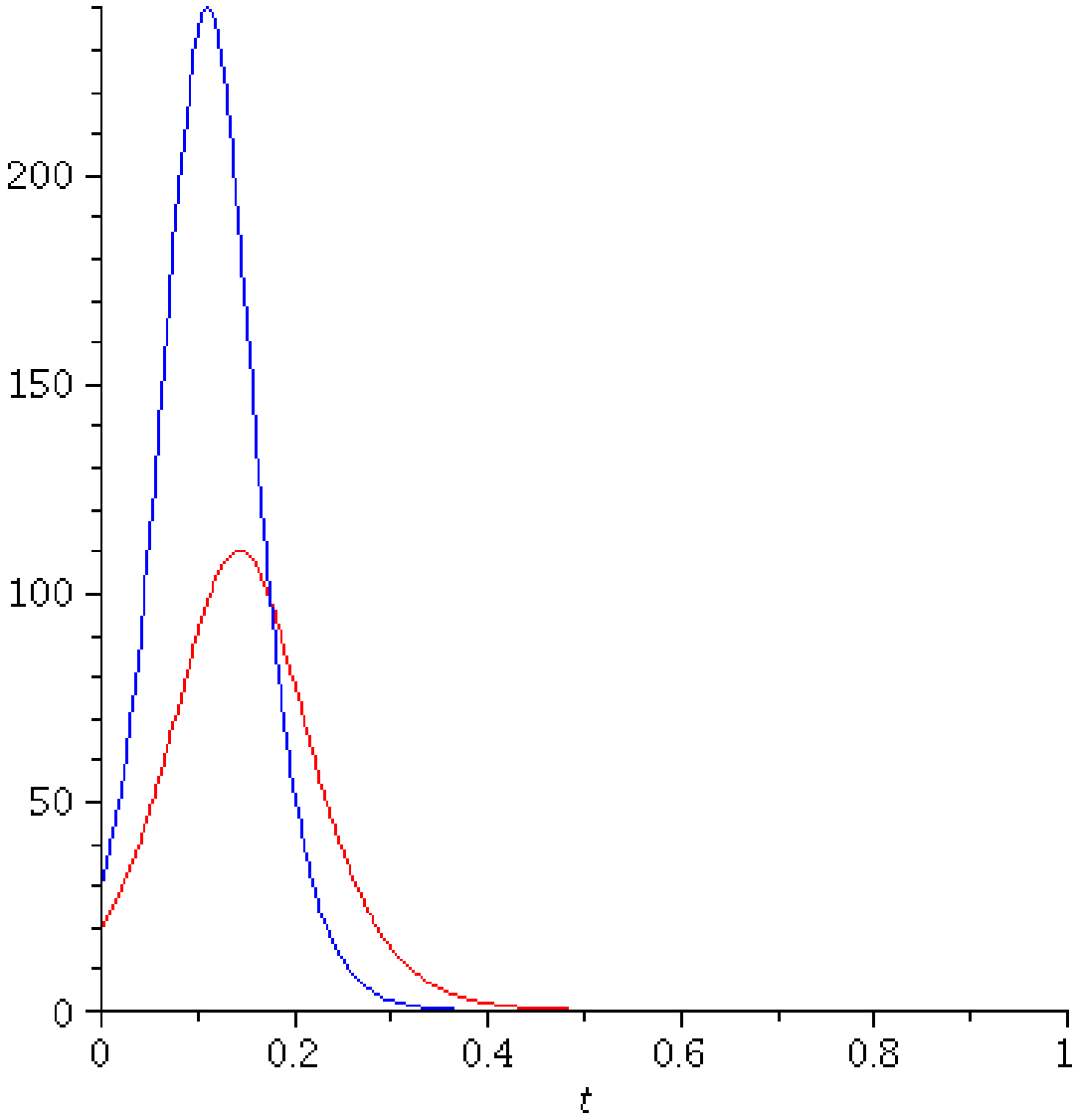

Figure 2.1: Epidemic curve with $n = 20$ (red) and for $n = 30$ (blue) for $0 \leq t \leq 1$.

As in the deterministic case in section 2.1, let us assume that there are n susceptibles and 1 infective at the start of the epidemic. Let $X(t)$ represent the random variable that describes the number of susceptibles still uninfected at time t. Next, let $p_r(t)$ denote the probability that $X(t)$ takes the value r, i.e.

$$P\{X(t) = r\} = p_r(t).$$

At time t, there are $X(t)$ susceptibles and $n - X(t) + 1$ infectives. Here, only one type of transition is possible, namely the occurrence of a new infection and a reduction of one unit in

the number of susceptibles. Let us assume that the chance of a new infection in a short time interval is proportional to the product of susceptibles, infectives, and the length of that time interval. The chance of an infection in Δt is then given by $\beta X(t)(n-X(t)+1)\Delta t$, where β is the constant of proportionality called the *contact rate*. Again for the sake of simplicity, we change the time scale to $\tau = \beta t$. Accordingly, the chance becomes $X(t)(n-X(t)+1)\Delta\tau$. Denote by $p_r(\tau)$ the probability that there are still r susceptibles remaining uninfected at time τ. At time $\tau+\Delta\tau$, there can be only one of the two possible scenarios: there are either $r+1$ susceptibles at τ followed by a new infection with probability $(r+1)(n-r)\Delta\tau$, or r susceptibles at τ followed by no infection with probability $1-r(n-r+1)\Delta\tau$. Hence, $p_r(\tau+\Delta\tau)$, the probability of r susceptibles remaining at time $\tau+\Delta\tau$, is given by

$$p_r(\tau+\Delta\tau) = (r+1)(n-r)\Delta\tau p_{r+1}(\tau) + [1-r(n-r+1)\Delta\tau]p_r(\tau). \tag{2.12}$$

Using (2.12), we obtain the differential-difference equation,

$$\begin{aligned}\frac{dp_r(\tau)}{d\tau} &= \lim_{\Delta\tau\to 0}\frac{p_{r+1}(\tau+\Delta\tau)-p_r(\tau)}{\Delta\tau} \\ &= (r+1)(n-r)p_{r+1}(\tau) - r(n-r+1)p_r(\tau)\end{aligned} \tag{2.13}$$

where $0 \le r \le n-1$. For $r=n$ at $\tau+\Delta\tau$, there are n susceptibles at τ followed by no new infection with probability $1-n\Delta\tau$. Hence, $p_n(\tau+\Delta\tau)$ is given by

$$p_n(\tau+\Delta\tau) = (1-n\Delta\tau)p_n(\tau) \tag{2.14}$$

and consequently, we obtain the first order linear differential equation

$$\frac{dp_n(\tau)}{d\tau} = \lim_{\Delta\tau\to 0}\frac{p_n(\tau+\Delta\tau)-p_n(\tau)}{\Delta\tau} = -np_n(\tau). \tag{2.15}$$

The equations (2.13) and (2.15) are now combined to a single equation

$$\frac{dp_r(\tau)}{d\tau} = (r+1)(n-r)p_{r+1}(\tau) - r(n-r+1)p_r(\tau) \tag{2.16}$$

where $0 \le r \le n$. At $\tau = 0$, there is only one infective, so we have the initial condition $p_n(0) = 1$.

One may attempt to solve the equation (2.16) successively similarly to that of the Poisson distribution, as seen in section 1.1. The first few are found to be

$$\begin{aligned}p_n(\tau) &= e^{-n\tau} \\ p_{n-1}(\tau) &= \frac{n}{n-1}[1-e^{-(n-1)\tau}]e^{-(n-1)\tau} \\ p_{n-2}(\tau) &= \frac{2n}{2n-5}e^{-(n-1)\tau} - \frac{2n}{n-4}e^{-2(n-1)\tau} + \frac{2n(n-1)}{(n-4)(2n-5)}e^{-3(n-1)\tau}\end{aligned}$$

It is not easy to see any pattern that may lead us to a general form of the solution, so, unfortunately, solving the equation successively is not an option in this case. As an alternative, we now attempt to solve (2.16) using the generating function method, as seen in section 1.1. Indeed solving differential equations by generating functions is a well-known method, particularly in physics. See [4] and volume III of [3] for details. Let us consider the probability generating function

$$P(x,r) = \sum_{r=0}^{n} p_r(\tau) x^r. \tag{2.17}$$

Multiplying (2.16) by x^r and summing it over r from $r = 0$ to n, we obtain the partial differential equation

$$\frac{\partial P}{\partial \tau} = (1-x)\left[n\frac{\partial P}{\partial x} - x\frac{\partial^2 P}{\partial x^2}\right] \tag{2.18}$$

The initial condition is given by $P(x,0) = x^n$ since $p_n(0) = 1$.

We now attempt to solve (2.18) using the typical separation of variables method. Let us assume that $P(x,\tau) = X(x)T(\tau)$. Putting it back into (2.18) and separating the equation into two parts that depend on τ and x, respectively.

$$\frac{T'}{T} = \frac{n(1-x)X' - x(1-x)X''}{X}.$$

As the usual argument goes, the only possibility that the left-hand side and the right-hand side are always the same regardless of two independent variables τ and x is when the two quantities are the same constant. Let us set the constant to be $-\lambda$ with $\lambda > 0$. The reason for choosing a negative constant is clear. Otherwise, we have a stationary solution ($\lambda = 0$) or have a divergent solution for $\tau \to \infty$ ($\lambda < 0$). This results in a set of two linear ordinary differential equations

$$T' = -\lambda T \tag{2.19}$$

$$x(x-1)X'' + [-nx+n]X' + \lambda X = 0. \tag{2.20}$$

The solution of (2.19) is $T(\tau) = e^{-\lambda\tau}$. The equation (2.20) resembles the hypergeometric equation (1.10). In fact, it can be viewed as the hypergeometric equation with $\lambda = ab$, $c = -n$ and $a+b = -n-1$. However, as we have seen in section 1.2, non-positive integers are not permitted for possible values of c. This obstacle may be overcome by assuming that n is not an integer. We replace n by $N = n+\varepsilon$, where ε is a small positive real number. Since N is not an integer, our new equation after replacing n by N will no longer pose an issue. One then hopes to retrieve the solution of the original equation by taking the limit $\varepsilon \to 0$.

This kind of technique is well-known and familiar in physics. It is called the *perturbation method* and was initially introduced by the great American physicist Richard Feynman. We now consider the equation

$$x(x-1)X'' + [-Nx+N]X' + \lambda X = 0 \tag{2.21}$$

Then $X(x) = {}_2F_1(a,b,c;x)$, where $\lambda = ab$, $c = -N$, $a+b = -N-1$, is a solution of (2.21). Since the probability generating function $P(x,\tau)$ is a polynomial in x of degree at most n, $X(x)$ must be a polynomial in x of degree at most n as well. This requires that a say[1] is a negative integer $a = -j$ with $0 \le j \le n$. Consequently, $b = j-N-1$ and the eigenvalues λ_j are given by $\lambda_j = j(N-j+1)$, $0 \le j \le n$. The general solution $P(x,\tau)$ can be then written as

$$P(x,\tau) = \sum_{j=0}^{n} d_j e^{-j(N-j+1)\tau} {}_2F_1(-j, j-N-1, -N; x). \tag{2.22}$$

The coefficients d_j are still to be determined. From the initial condition $P(x,0) = x^n$ along with (2.22), we have

$$x^n = \sum_{j=0}^{n} d_j e^{-j(N-j+1)\tau} {}_2F_1(-j, j-N-1, -N; x). \tag{2.23}$$

It is helpful to know that *Jacobi polynomials* can be represented in terms of hypergeometric functions, and they satisfy a particular orthogonality condition. (See volume II, chapter 10 of [3] for details.) Jacobi polynomials are denoted by $P_j^{\mu,\nu}(y)$ where $j \ge 0$ and $\mu, \nu > -1$. As mentioned above, The Jacobi polynomial $P_j^{\mu,\nu}(y)$ can be given in terms of a hypergeometric function as

$$P_j^{\mu,\nu}(y) = \binom{j+\mu}{j} {}_2F_1\left(-j, j+\mu+\nu+1, \mu+1; \frac{1-y}{2}\right) \tag{2.24}$$

In our case, we take the values $x = \frac{1-y}{2}$, $\mu = -N-1$ and $\nu = -1$. Although the standard results on Jacobi polynomials are restricted to $\mu, \nu > -1$, they hold more generally. In terms of Jacobi polynomials, (2.23) can be written as

$$\left(\frac{1-y}{2}\right)^n = \sum_{k=0}^{n} d_k \frac{P_k^{\mu,\nu}(y)}{\binom{k+\mu}{k}}. \tag{2.25}$$

Jacobi polynomials satisfy the following orthogonality condition

$$\int_{-1}^{1} w(y) P_j^{\mu,\nu}(y) P_k^{\mu,\nu}(y) dy = \begin{cases} \frac{2^{\mu+\nu+1}\Gamma(j+\mu+1)\Gamma(j+\nu+1)}{(2j+\mu+\nu+1)j!\Gamma(j+\mu+\nu+1)} & \text{if} \quad j = k, \\ 0 & \text{if} \quad j \ne k \end{cases} \tag{2.26}$$

[1] One may choose b to a negative integer, but that would be inconvenient with Jacobi polynomials, as we will see later.

where the weight function, $w(y)$ is chosen to be

$$w(y) = (1-y)^{\mu}(1+y)^{\nu}. \tag{2.27}$$

Furthermore, we have

$$\int_{-1}^{1} w(y)f(y)P_j^{\mu,\nu}(y)dy = \frac{2^{-j}}{j!}\int_{-1}^{1} f^{(j)}(y)w(y)\left(1-y^2\right)^j dy \tag{2.28}$$

Here, $f^{(j)}(y)$ denotes the j-th order derivative of $f(y)$. with $f(y) = \left(\frac{1-y}{2}\right)^n$ and $(x^n)^{(j)} = n(n-1)(n-2)\cdots(n-j+1)x^{n-j}$. Multiplying (2.23) by $w(y)P_j^{\mu,\nu}(y)$ and integrating from -1 to 1 results in the LHS

$$\int_{-1}^{1} w(y)\left(\frac{1-y}{2}\right)^n P_j^{\mu,\nu}(y)dy = \frac{(-1)^j 2^{\mu+\nu+1} n!\Gamma(n+\mu+1)\Gamma(j+\nu+1)}{j!(n-j)!\Gamma(n+j+\mu+\nu+2)}$$

and in the RHS

$$\sum_{k=0}^{n} \frac{d_k}{\binom{k+\mu}{k}}\int_{-1}^{1} w(y)P_k^{\mu,\nu}(y)P_j^{\mu,\nu}(y)dy = \frac{d_j}{\binom{j+\mu}{j}}\frac{2^{\mu+\nu+1}\Gamma(j+\mu+\nu)\Gamma(j+\nu+1)}{(2j+\mu+\nu+1)j!\Gamma(j+\mu+\nu+1)}$$

due to the orthogonality condition (2.26). Setting the LHS and the RHS equal and solving it for d_j, we have

$$\begin{aligned} d_j &= \frac{(-1)^j n!(2j+\mu+\nu+1)\Gamma(n+\mu+1)\Gamma(j+\mu+\nu+1)}{(n-j)!\Gamma(j+\mu+1)\Gamma(n+j+\mu+\nu+2}\frac{(j+\mu)!}{j!\mu!} \\ &= \frac{(-1)^j n!(2j+\mu+\nu+1)(n+\mu)\cdots(\mu+1)}{j!(n-j)!(n+j+\mu+\nu+1)\cdots(j+\mu+\nu+1)} \end{aligned}$$

Putting $\mu = -N-1$ and $\nu = -1$, d_j can be written as

$$d_j = \frac{(-1)^j(N-2j+1)n!N!}{j!(n-j)!(N-n)!(N-j+1)\cdots(N-j-n+1)} \tag{2.29}$$

Definition 2.2.1 (The Stochastic Mean $\mu(\tau)$)**.** The mean (or expected value) of the probabilities $p_r(\tau)$

$$\left.\frac{\partial P(x,\tau)}{\partial x}\right|_{x=1} = \sum_{r=1}^{n} r p_r(\tau)$$

is called the *stochastic mean* and is denoted by $\mu(\tau)$.

Differentiating (2.22) with respect to x at $x=1$, the stochastic mean is calculated to be

$$\begin{aligned}\mu(\tau) &= \left.\frac{\partial P(x,\tau)}{\partial x}\right|_{x=1} \\ &= \sum_{j=1}^{n} d_j e^{-j(N-j+1)\tau} \frac{-j(j-N-1)}{-N} {}_2F_1(-j+1, j-N, -N+1; 1) \\ &= \sum_{j=1}^{n} e^{-j(N-j+1)\tau} \frac{(-1)^{j+1}(N-2j+1)n!}{(n-j)!(N-n)(N-n-1)\cdots(N-n-j+1)}\end{aligned} \tag{2.30}$$

using (1.14), (2.29) and

$${}_2F_1(-j+1, j-N, -N+1; 1) = \frac{\Gamma(N-j+1)\Gamma(j)}{\Gamma(N)}$$

This is a proper place where we take the limit $\varepsilon \to 0$. However, we quickly notice the presence of $N-n$ in the denominator of the last expression for $\mu(\tau)$ in (2.30), which appears to lead to the divergence of $\mu(\tau)$ as $\varepsilon \to 0$. A closed expression for $\mu(\tau)$ in the case $N=n$ was first obtained by H. W. Haskey in [5] through extremely complex algebra computations involving partial fraction expansions of the Laplace transform of probabilities. The result is

$$\mu(\tau) = \sum_{j=1}^{\frac{n}{2}} \frac{n!}{(n-j)!(j-1)!} \left\{ (n-2j+1)^2\tau + 2 - (n-2j+1) \sum_{u=j}^{n-j} \frac{1}{u} \right\} e^{-j(n-j+1)\tau} \tag{2.31}$$

if n is even and

$$\mu(\tau) = \sum_{j=1}^{\frac{n-1}{2}} \frac{n!}{(n-j)!(j-1)!} \left\{ (n-2j+1)^2\tau + 2 - (n-2j+1) \sum_{u=j}^{n-j} \frac{1}{u} \right\} e^{-j(n-j+1)\tau} + \frac{n!}{\left[\left(\frac{n-1}{2}\right)!\right]^2} \tag{2.32}$$

if n is odd. In [1] and [2], Bailey claims to have obtained a finite limit of (2.30) when $\varepsilon \to 0$ and have arrived at the same result as that of Haskey's, but he offers no details. Despite numerous attempts and much effort, we were not able to confirm Bailey's claim. However, there appears to be an indication that Haskey's result may not be entirely accurate, at least with the case when n is odd. This will be discussed in the next section, and we conclude this section by introducing the notion of the epidemic curve in the stochastic case.

Definition 2.2.2. The *epidemic curve* $\omega(\tau)$ is defined to be negative of the rate of change of the mean probability $\mu(\tau)$, namely

$$\omega(\tau) = -\frac{d\mu(\tau)}{d\tau}$$

2.3 Conclusions and Future Study

At the end of the previous section, we mentioned that there appears to be an indication that Haskey's result may not be entirely correct. First, figure 2.2 shows the epidemic curve for $n = 30$.

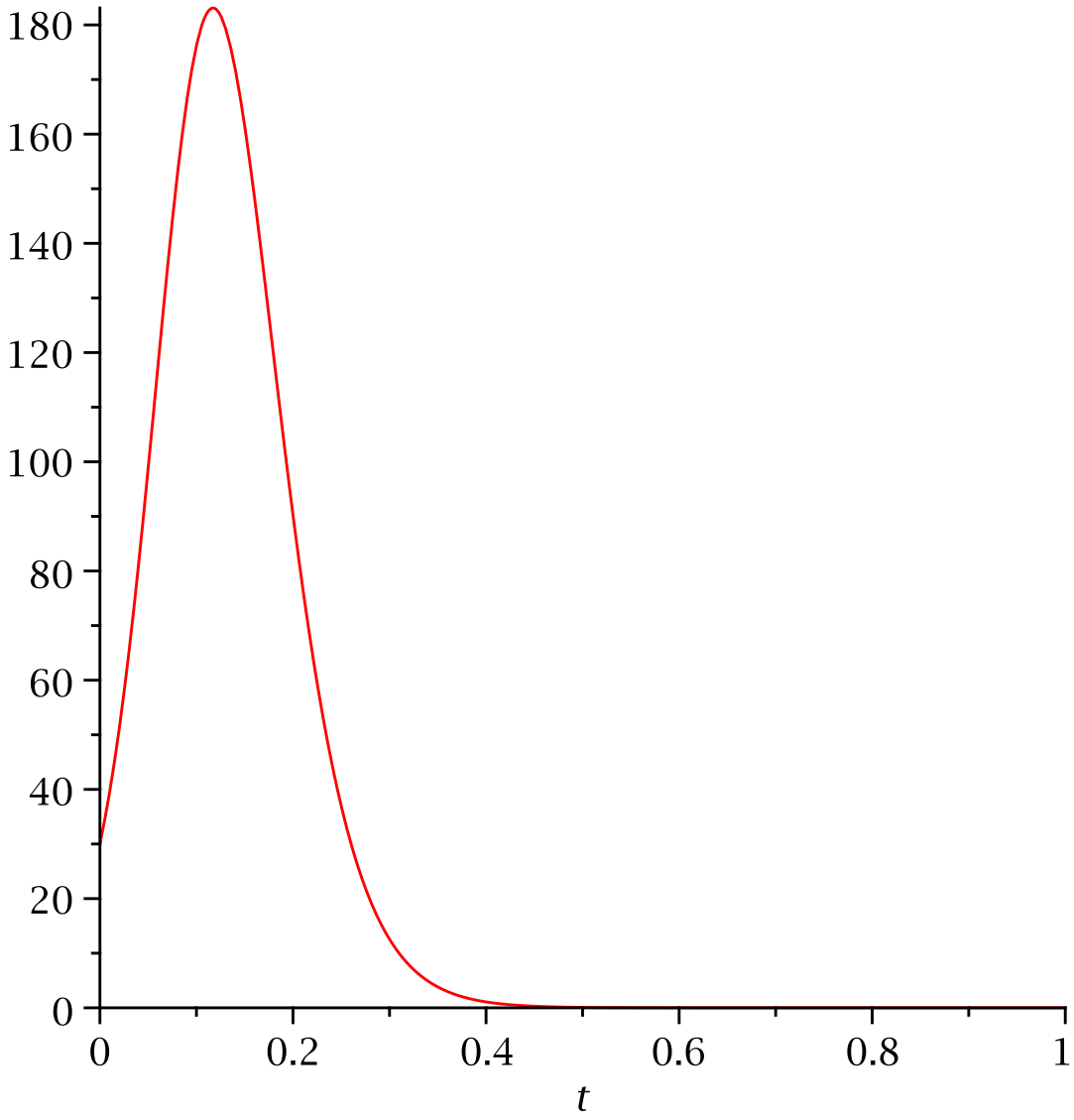

Figure 2.2: Epidemic curve with $n = 30$ for $0 \leq t \leq 1$.

The figure was obtained by using (2.31). Its pattern is similar to the deterministic case we have seen in figure 2.1. To see that better, we put epidemic curves from deterministic and stochastic models together for $n = 30$ in figure 2.3.

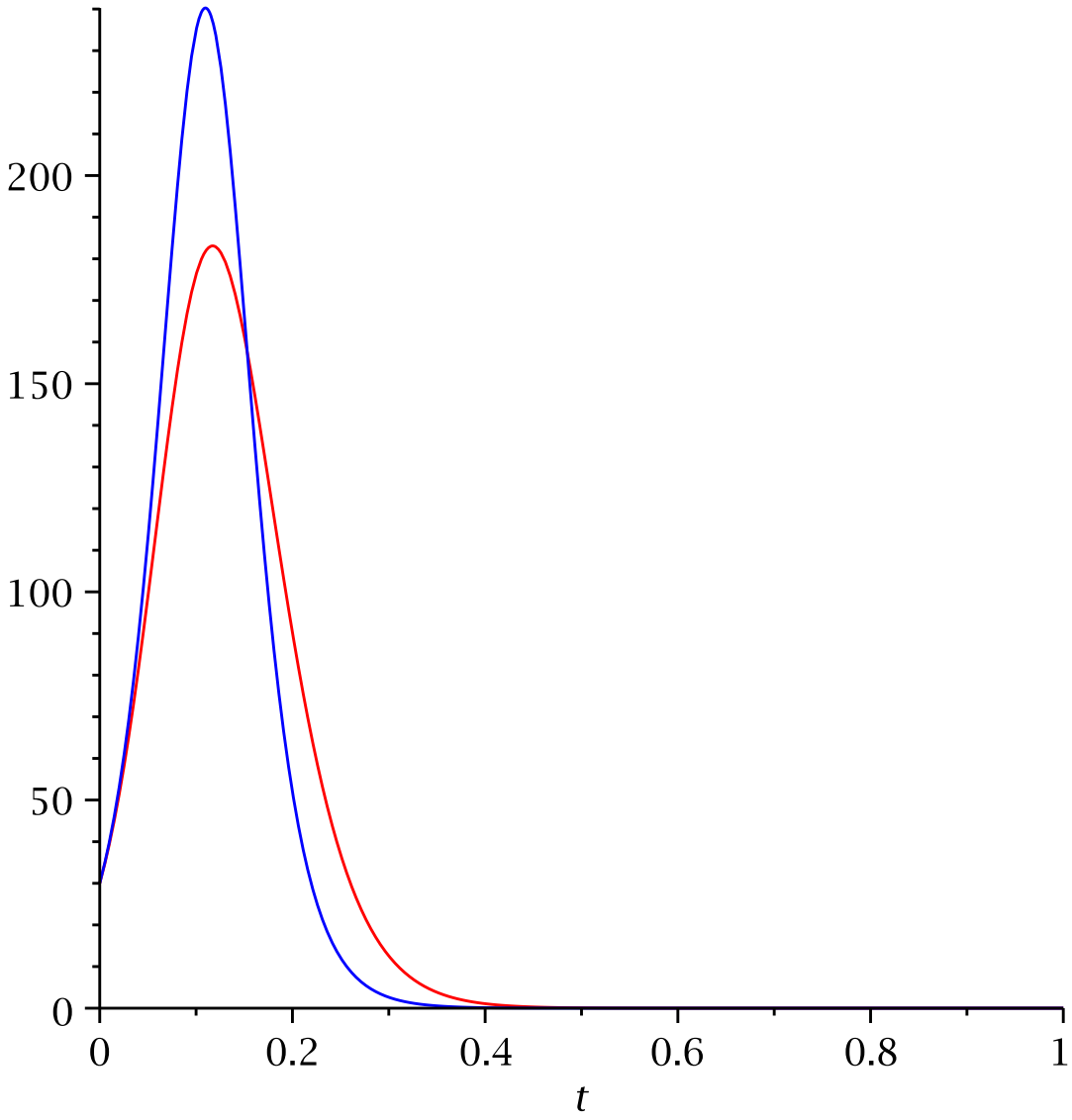

Figure 2.3: Deterministic epidemic curve (blue) and stochastic epidemic curve (red) with $n = 30$ for $0 \leq t \leq 1$.

While the deterministic epidemic curve and stochastic epidemic curve do not represent the same quantity, there is a correlation between them, which is reflected in figure 2.3. We noticed an issue when we plotted the stochastic epidemic curve for an odd number of the population using (2.32), as shown in figure 2.4.

The pattern of the epidemic curve does not look like that for the case n is even. This must not be the case, as the pattern of the epidemic curve must not be dependent on whether n is even or odd. In fact, it is not right at all, as the values of the epidemic curve must not be

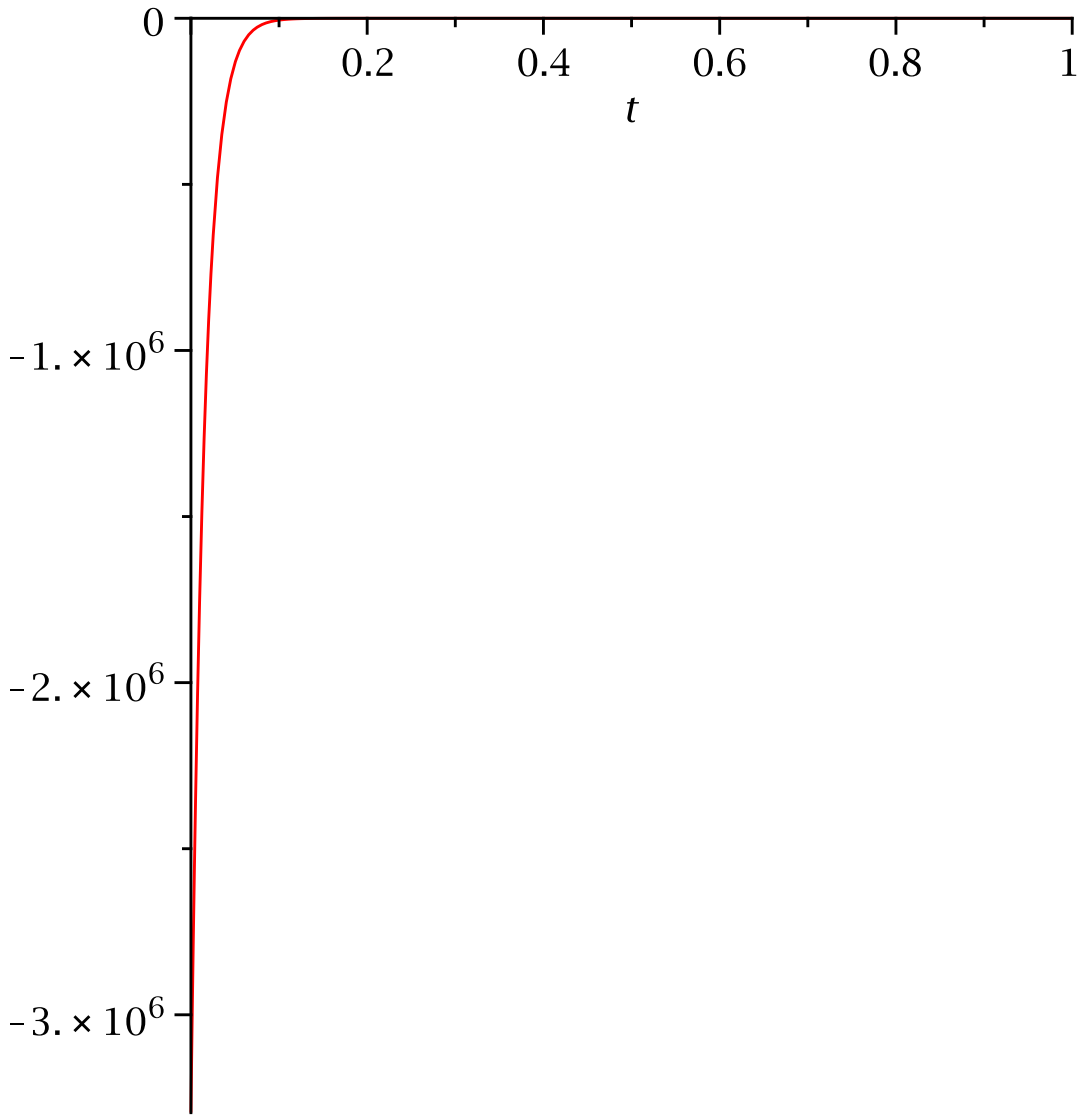

Figure 2.4: Epidemic curve with $n = 15$ for $0 \leq t \leq 1$.

negative. This hints that Haskey's result may not be correct, at least for the case n is odd.

The stochastic model introduced by Bailey is an intelligent approach. However, as we have seen, even for a simple stochastic model, the computations involved are so complex and challenging that it does not seem to be a practically viable approach for developing more generalized models. Also, even if Haskey's result is correct, it cannot predict when the rate of change of the infectives peaks. No matter how fancy the model theoretically appears, it would be practically useless if it lacks tractable computability and predictability. Based

on our assessment of Bailey's model, we argue that a better approach to stochastic modeling of epidemics is called for. We conclude this thesis by mentioning a couple of related future projects that we will continue to explore:

1. It turns out even when $c = -n$ where $n = 0, 1, 2, \cdots$ the hypergeometric equation (1.10) has a solution

$$y(x) = x^{n+1}{}_2F_1(a+n+1, b+n+1, n+2, x)$$

(See volume I of [3].) If we use this solution, we would not need to use the method of perturbation. We wonder why Bailey did not use this solution. The reason could be that it might be challenging to find suitable orthogonal functions. Regardless, this route should be looked into.

2. One may introduce a different approach to stochastic modeling of epidemics by considering the equation

$$\frac{dx(t)}{dt} = -\beta(t)x(t)(n - x(t) + 1) \tag{2.33}$$

with $x(0) = a$. The infection rate is no longer constant but a function of t. It is not completely known but can be written as $\beta(t) = r(t) + \text{noise}$. We do not know the exact behavior of noise but only its probability distribution. In other words, equation (2.33) is a *stochastic differential equation*. Whether this approach by a stochastic differential equation would be promising for modeling epidemics is remained to be seen. However, we believe this route is worth a try because there is a treasure trove of well-developed tools for stochastic differential equations.

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