Shape of Sound

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Abstract

In this project, we model a vibrating drumhead. A vibrating drumhead can be modeled by the wave equation in polar coordinates.

1 Modeling of a Vibrating Drumhead

The motion of a vibrating drumhead can be described by the wave equation in polar coordinates

$$\text{PDE: } u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \right), \ 0 < r < 1. \tag{1}$$

Here we consider our drumhead as a unit circle (circle with radius 1). The solution $u = u(r, \theta, t)$ stand for the height of the drumhead from the plane (where $u = 0$). The rim of the drumhead must be tied down, so we naturally impose the boundary condition

$$\text{BC: } u(1, \theta, t) = 0, \ 0 < t < \infty. \tag{2}$$

In order to describe motion of the drumhead, we also need to specify the initial conditions

$$\text{ICs: } \begin{array}{l} u(r, \theta, 0) = f(r, \theta) \\ u_t(r, \theta, 0) = g(r, \theta) \end{array} \tag{3}$$

that are, respectively, the initial position and the initial velocity of the drumhead.

Separation of Variables Method

We solve the wave equation using the separation of variables method. We assume that $u(r, \theta, t)$ takes the form

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t). \tag{4}$$

This assumption allows us to reduce the wave equation (1) to three ordinary differential equations

$$T'' + \lambda^2 c^2 T = 0 \tag{5}$$
$$r^2 R'' + rR' + (\lambda^2 r^2 - n^2)R = 0 \tag{6}$$
$$\Theta'' + \mu\Theta = 0 \tag{7}$$

where λ and μ are nonzero constants. The equations (5) and (7) are simple harmonic oscillators. The equation (6) is called Bessel's equation. $R(r)\Theta(\theta)$ determines the shape of the drumhead while $T(t)$ determines the oscillatory motion of the drumhead.

The Angular Sturm-Liouville Problem

The function $\Theta(\theta)$ satisfies the Angular Sturm-Liouville Problem

$$\Theta'' = \mu\Theta$$
$$\Theta(0) = \Theta(2\pi) \text{ (Periodic BC)}$$

The periodic BC is imposed because we want Θ to be periodic with period 2π.

The eigenvalues must have the form

$$\mu = -n^2,\ n = 0, 1, 2, \cdots .$$

The corresponding eigenfunctions are

$$\Theta_0(\theta) = 1$$
$$\Theta_n^1(\theta) = \cos n\theta$$
$$\Theta_n^2(\theta) = \sin n\theta.$$

The Radial Sturm-Liouville Problem

The function $R(r)$ satisfies the Radial Sturm-Liouville Problem

$$r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0,\ 0 < r < 1 \text{ Bessel' equation)}$$
$$R(1) = 0$$
$$R(0) < \infty \text{ (physical condition)}$$

The Bessel's equation has two independent solutions

$$R_1(r) = J_n(\lambda r) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda r)^{2k+n}}{2^{2k+n} k!(n+k)!} \tag{8}$$

$$R_2(r) = Y_n(\lambda r) = \frac{2}{\pi} J_n(\lambda r)\left[\ln\left(\frac{\lambda r}{2}\right) + \gamma\right] + \frac{2^n}{\pi(\lambda r)^n} \sum_{k=0}^{\infty} \frac{\beta_{nk}}{2^{2k} k!} (\lambda r)^{2k} \tag{9}$$

Here γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right]$$

and β_{nk} are the numbers defined by

$$\beta_{nk} = \begin{cases} -(n-1-k)! & \text{if} \quad k \le n-1 \\ (-1)^{k-n-1} \frac{(h_{k-n}+h_k}{(k-n)!} & \text{if} \quad k > m-1 \end{cases}$$

where $h_p = \sum_{i=1}^{p} \frac{1}{i}$. The solution (8) and (9) are called, respectively, the n-th order Bessel function of the first kind and the n-th order Bessel function of the 2nd kind. The general solution $R(r)$ is then given by the linear combination

$$R(r) = AJ_n(\lambda r) + BY_n(\lambda r).$$

Since $Y_n(\lambda r)$ is not defined at $r = 0$, $B = 0$. By the BC $R(1) = 0$, we obtain

$$J_n(\lambda) = 0. \tag{10}$$

The solutions λ_{nm} of equation (10) are the eigenvalues. The corresponding radial eigenfunctions are then given by

$$R_{nm}(r) = J_n(\lambda_{nm} r). \tag{11}$$

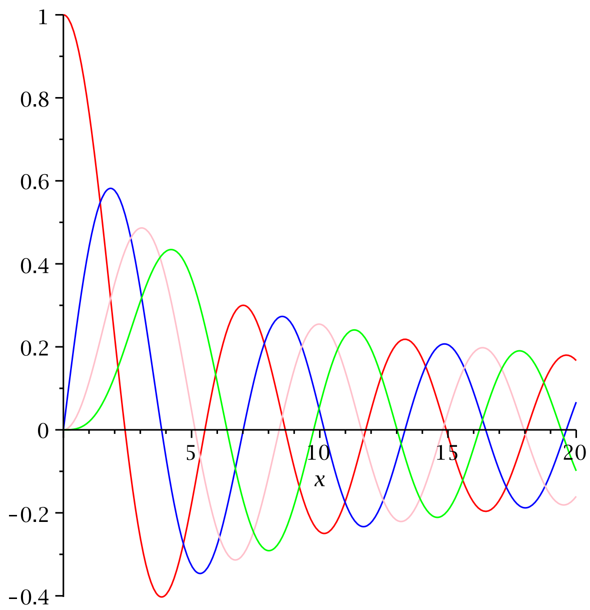


Figure 1: Bessel functions $J_0(x)$ (red), $J_1(x)$ (blue), $J_2(x)$ (pink), $J_3(x)$ (green) on $[0, 20]$

Oscillating Factors

The oscillating factors are determined by the equation

$$T'' + \lambda_{nm}^2 c^2 T = 0. \tag{12}$$

The solutions are

$$T_{nm}(t) = C\cos(\lambda_{nm}t) + D\sin(\lambda_{nm}t). \tag{13}$$

The Solution of the Vibrating Drumhead Problem

Finally the solution of our vibrating drumhead problem can be written as

$$u(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(\lambda_{nm}r)\cos(n\theta)[A_{nm}\cos(\lambda_{nm}t + B_{nm}\sin(\lambda_{nm}t)]. \tag{14}$$

Using the orthogonality of Bessel functions, we can determine the coefficients A_{nm} and B_{nm} as follows:

$$A_{0m} = \frac{1}{2\pi L_{0m}} \int_0^{2\pi} \int_0^1 f(r,\theta) J_0(\lambda_{0m} r) r dr d\theta, \ m = 1, 2, \cdots \tag{15}$$

$$A_{nm} = \frac{1}{\pi L_{nm}} \int_0^{2\pi} \int_0^1 f(r,\theta) J_n(\lambda_{nm} r) \cos(n\theta) r dr d\theta, \ n, m = 1, 2, \cdots \tag{16}$$

$$B_{0m} = \frac{1}{2\pi \lambda_{0m} L_{0m} c} \int_0^{2\pi} \int_0^1 g(r,\theta) J_0(\lambda_{0m} r) r dr d\theta, \ m = 1, 2, \cdots \tag{17}$$

$$B_{nm} = \frac{1}{\pi \lambda_{nm} L_{nm} c} \int_0^{2\pi} \int_0^1 g(r,\theta) J_n(\lambda_{0m} r) \cos(n\theta) r dr d\theta, \ n, m = 1, 2, \cdots \tag{18}$$

where

$$L_{nm} = \int_0^1 J_n(\lambda_{nm} r)^2 r dr, \ n = 0, 1, 2, \cdots, \ m = 1, 2, \cdots. \tag{19}$$

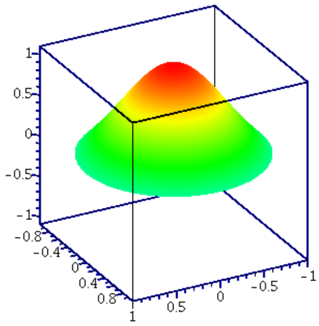
Example

We solve an explicit model of a drumhead with $c = 1$, $f(r,\theta) = J_0(2.4r) + 0.1J_0(5.52r)$, and $g(r,\theta) = 0$. Pictures in the last page show some still images of the motion of the resulting drumhead.

The animation of this drumhead can be viewed at `http://www.math.usm.edu/lee/drumhead.gif`.

References

[1] David Betounes, Partial Differential Equations for Computational Science with Maple and Vector Analysis, Telos, 1998
[2] Stanley J. Farlow, Partial Differential Equations for Scientists and Engineers, Dover, 1993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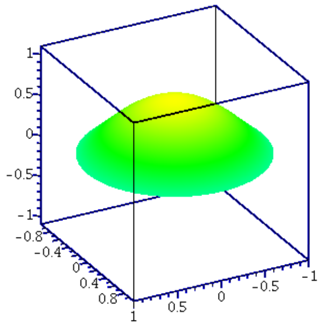
(a)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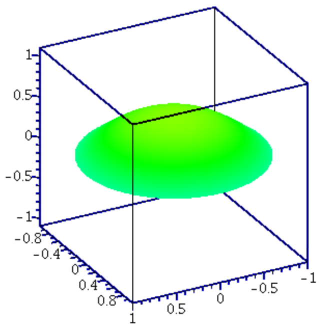
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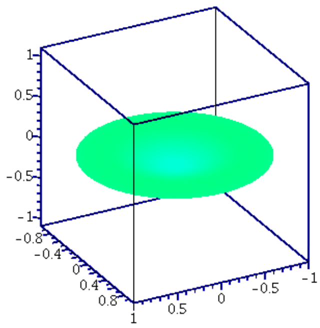
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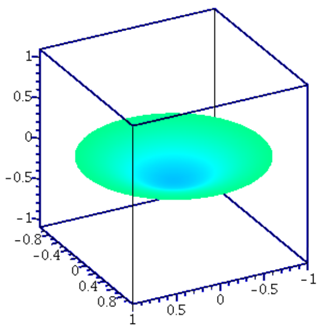
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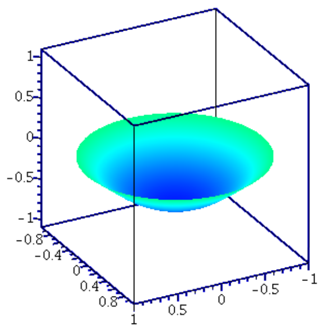
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