What is a Quantum Computer?

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Outline

1. Quantum Mechanics

2. Quantum Computing

3. An Example of a Quantum Circuit

4. Simulating a Quantum Computer

The Hilbert Space of Quantum States

- The state of a quantum mechanical system is described by a vector in a Hilbert space.

- State vectors are written by the notation $|\psi\rangle$. $|\psi\rangle$ is called a *ket vector*.

- The dual vector (a 1-form) of a ket vector $|\psi\rangle$ is denoted by $\langle\psi|$ and called a *bra vector*.

- The squared norm $\langle\psi|\psi\rangle$ is called a *braket*. In quantum mechanics, $\langle\psi|\psi\rangle$ measures the probability of getting a particle in the state $|\psi\rangle$.

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2-State System

- A *2-state system* (the basic building block of a quantum memory register) can, by definition, be in one of two possible states.

- Its state vector $|\psi\rangle$ has exactly two componenets:

$$|\psi\rangle = \omega_0 |\psi_0\rangle + \omega_1 |\psi_1\rangle = \omega_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \omega_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix},$$

  where $\omega_0, \omega_1 \in \mathbb{C}$ and the eigenstates (pure states) $|\psi_0\rangle, |\psi_1\rangle$ form an orthonormal basis for the Hilbert space.

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Schrödinger's Cat
Erwin Schrödinger's Gedankenexperiment

Figure: Schrödinger's Cat

Schrödinger's Cat
Compound States

Figure: Half-Life and Half-Death

Spin-$\frac{1}{2}$ Particles

- *Spin* is an intrinsic property of particles obtained from the quantization of angular momentum of particles.

- *Fermions* such as electrons have spin-$\frac{1}{2}$. In fact electron was the first particle of which spin property was observed experimentally.

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Spin Operator

- The spin projection operator S_z

$$S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right).$$

- S_z has two eigenvalues $\pm\frac{\hbar}{2}$, which correspond to the eigenstates

$$\left(\begin{array}{c} 1 \\ 0 \end{array}\right) = |+\frac{1}{2}\rangle = |\uparrow\rangle,$$

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Bloch Sphere

- *Bloch sphere* is a geometrical representation of 2-state quantum mechanical system.

- A state vector $|\psi\rangle$ has a representation in Bloch sphere as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle,$$

where $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$.

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Bloch sphere
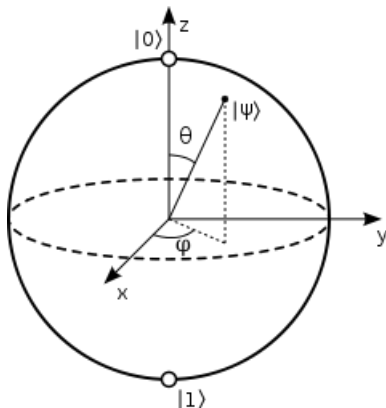## The Visualization of Bloch Sphere

Figure: Bloch Sphere

Schrödinger's Equation
The Evolution of a State Vector

- If not observed, the quantum mechanical system will undergo smooth continuous evolution governed by Schrödinger's equation

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = \hat{H}(t)|\psi(t)\rangle,$$

where $\hat{H}(t)$ is the Hamiltonian operator.

Schrödinger's Equation
Continued

- Equivalently in terms of the temporal evolution of the amplitudes, we have

$$i\hbar\frac{d\omega_i}{dt} = \sum_j H_{ij}(t)\omega_j(t).$$

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From Bits to Qubits

- In a (hypothetical) quantum computer, each bit can be represented by the pure state of a 2-state quantum system such as the spin state of spin-$\frac{1}{2}$ particle.

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State of a Quantum Memory Register

- Each *qubit* is described by the spin state of a single 2-state system.

- Fermions such as electrons follow *Fermi-Dirac statistics*, i.e. only one particle can occupy a quantum state at a given time.

- A memory register consists of many such 2-state systems. The question is how do we describe its state?

- The general state of a *2-qubit memory register* is given by the direct product

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$$|\psi^{(2)}\rangle = \omega_0^{(2)}|\psi_0^{(2)}\rangle + \omega_1^{(2)}|\psi_1^{(2)}\rangle = \begin{pmatrix} \omega_0^{(2)} \\ \omega_1^{(2)} \end{pmatrix},$$

where $|\psi_0^{(1)}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\psi_1^{(1)}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $|\psi_0^{(2)}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$|\psi_1^{(2)}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

State of a Quantum Memory Register
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$$|\psi^{(1,2)}\rangle = |\psi^{(1)}\rangle \otimes |\psi^{(2)}\rangle = \begin{pmatrix} \omega_0^{(1)}\omega_0^{(2)} \\ \omega_0^{(1)}\omega_1^{(2)} \\ \omega_1^{(1)}\omega_0^{(2)} \\ \omega_1^{(1)}\omega_1^{(2)} \end{pmatrix} = \begin{pmatrix} \omega_{00} \\ \omega_{01} \\ \omega_{10} \\ \omega_{11} \end{pmatrix}.$$

State of a Quantum Memory Register
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- The new eigenstates of the composite syste are $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, where

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Why Quantum Memory Register is big deal?

- The generalization to an *n-qubit quantum memory register* is straightforward.
- Quantum memory register can store an exponential amount of classical information in only a polynomial number of qubits by exploiting the principle of superposition.
- For example, the table shows two classical memory registers storing complementary sequences of bits.

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Table: Two classical memory registers

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Why Quantum Memory Register is big deal?
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- However a single quantum memory register can store both sequences simultaneously in an equally weighted superposition of the two states representing each classical input, i.e. the quantum memory register stores the state

$$\frac{1}{\sqrt{2}}(|1011001\rangle + |0100110\rangle).$$

Unitary Evolution
A Pathway to Reversible Computing

- If the Hamiltonian $\hat{H}$ is time-independent and the computer is started off with its memory register in the state $|\psi(0)\rangle$, then the solution of the Schrödinger equation at a given time t is given by

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle = \hat{U}(t)|\psi(0)\rangle.$$

- $\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$ is called the *evolution operator*. Since $\hat{H}$ is Hermitian, $\hat{U}(t)$ is a unitary matrix, i.e. $\hat{U}(t)\hat{U}^{\dagger}(t) = I$ or equivalently $\hat{U}^{-1}(t) = U^{\hat{\dagger}}(t)$.

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A Quantum Computer is a Reversible Computer

- The evolution operator being unitary has a very important implication. This means that a quantum computer is a *reversible computer*.

- A reversible computer does not generate heat while performing computations since it results in no increase of entropy. So a quantum computer can be a truly energy efficient computer.

- According to Landauer's principle, any logically irreversible manipulation of information, such as erasing bits of information, must be accompanied by an increase of entropy. Each bit of lost information will result in the release of an amount $kT \ln 2$ of heat, where k is the Boltzmann constant and T is the absolute temperature of the circuit.

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$\sqrt{\text{NOT}}$ Gate

- $\sqrt{\text{NOT}}$ is a quantum logical gate.

- Two consecutive applications of $\sqrt{\text{NOT}}$ gate results NOT gate.

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  where $\cdot$ denotes the matrix multiplication.

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$\sqrt{\text{NOT}}$ Gate as a Unitary Matrix

- Define the action of the $\sqrt{\text{NOT}}$ gate as

$$\sqrt{\text{NOT}} = \left[\begin{array}{cc} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{array} \right].$$

- $\sqrt{\text{NOT}}\sqrt{\text{NOT}}^{\dagger} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$, so $\sqrt{\text{NOT}}$ is unitary, i.e. $\sqrt{\text{NOT}}$ gate is reversible.

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The Quantum Nature of $\sqrt{\text{NOT}}$ Gate

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$$|0\rangle \to \left(\frac{1}{2} + \frac{i}{2}\right)|0\rangle + \left(\frac{1}{2} - \frac{i}{2}\right)|1\rangle$$

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- $\sqrt{\text{NOT}}$ gate has the effect of taking states representing classical bits, $|0\rangle$ and $|1\rangle$, into mixed states representing qubits, $\omega_0 |0\rangle + \omega_1 |1\rangle$.

- The inputs for $\sqrt{\text{NOT}}$ gate need not be pure states $|0\rangle$ and $|1\rangle$. Any superposition of $|0\rangle$ and $|1\rangle$ can be an input for $\sqrt{\text{NOT}}$ gate. This aspect is particularly important when we connect two $\sqrt{\text{NOT}}$ gates back to back to form a circuit for NOT.

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The Size of the Memory Register

- We need two sets of particles to form the Memory Register:
  1. One set of particles to record the position of the cursor qubit that keeps track of the progress of computation.
  2. The other set of particles is used to record the answer to the computation on which the machine is working, i.e. the evolving state of computation.

- If there are k gates and m input quibits, the size of the memory register is $m + k + 1$ to simulate the entire computer.

- For $\sqrt{\text{NOT}} \cdot \sqrt{\text{NOT}}$ circuit, $k = 2$ and $m = 1$ so $m + k + 1 = 4$.

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$\sqrt{\text{NOT}}$ with a 4-Qubit Register

- The complete state of the computer (cursor qubits and answer qubits) is represented as a $2^4 \times 1$ column vector, i.e. the complete memory register for the quantum computer is composed of 4 qubits.

- Since $\sqrt{\text{NOT}}$ acts on a single qubit, we need to extend it to $\sqrt{\text{NOT}}[4,4]$:

$$\sqrt{\text{NOT}}[4,4] \equiv \hat{1} \otimes \hat{1} \otimes \hat{1} \otimes \sqrt{\text{NOT}}.$$

- $\sqrt{\text{NOT}}[4,4]$ is a $2^4 \times 2^4$ unitary matrix.

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$\sqrt{\text{NOT}}$ with a 4-Qubit Register

- The complete state of the computer (cursor qubits and answer qubits) is represented as a $2^4 \times 1$ column vector, i.e. the complete memory register for the quantum computer is composed of 4 qubits.

- Since $\sqrt{\text{NOT}}$ acts on a single qubit, we need to extend it to $\sqrt{\text{NOT}}[4, 4]$:

$$\sqrt{\text{NOT}}[4, 4] \equiv \hat{1} \otimes \hat{1} \otimes \hat{1} \otimes \sqrt{\text{NOT}}.$$

- $\sqrt{\text{NOT}}[4, 4]$ is a $2^4 \times 2^4$ unitary matrix.

$\sqrt{\text{NOT}}[4,4]$

$$\sqrt{\text{NOT}}[4,4] = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} & & & & & & \\ \frac{1-i}{2} & \frac{1+i}{2} & & & & & & \\ & & \frac{1+i}{2} & \frac{1-i}{2} & & & & \\ & & \frac{1-i}{2} & \frac{1+i}{2} & & & & \\ & & & & \ddots & & & \\ & & & & & \frac{1+i}{2} & \frac{1-i}{2} & \\ & & & & & \frac{1-i}{2} & \frac{1+i}{2} & \\ & & & & & & & \frac{1+i}{2} & \frac{1-i}{2} \\ & & & & & & & \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

The Creation and Annihilation Operators

- The creation and annihilation operators are used to move the cursor forwards and backwards.

- Moving the cursor from the ith site to $(i+1)$th site and coupling it to the application of the $(i+1)$th gate operation is done by

$$c_{i+1} \cdot a_i \cdot \hat{M_{i+1}}.$$

- The creation operator c on a single spin state is $c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The annihilation operator a acting on a single spin state is $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

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The Creation and Annihilation Operators
Continued

- As a 4-qubit, the creation operator acting on the second of four qubits is given by

$$
c_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \otimes \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right] \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \otimes \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right].
$$

The Hamiltonian Operator

- Define the Hamiltonian operator $\hat{H}$ to be

$$\hat{H} = \sum_{i=0}^{k} c_{i+1} \cdot a_i \cdot \hat{M}_{i+1} + (c_{i+1} \cdot a_i \cdot \hat{M}_{i+1})^{\dagger}.$$

- In $\sqrt{\text{NOT}} \cdot \sqrt{\text{NOT}}$ circuit,

$$\hat{H} = c_1 \cdot \sqrt{\text{NOT}} + c_2 \cdot a_1 \cdot \sqrt{\text{NOT}} \cdot \sqrt{\text{NOT}}$$
$$+ (c_1 \cdot \sqrt{\text{NOT}} + c_2 \cdot a_1 \cdot \sqrt{\text{NOT}} \cdot \sqrt{\text{NOT}})^{\dagger}.$$

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Running the Quantum Computer

- The Unitary Evolution Operator $\hat{U}(t)$:

$$\hat{U}(t) = \exp\left(-\frac{i}{\hbar}\hat{H}t\right).$$

- Once a particular initial state of the memory register $|\psi(0)\rangle$ is selected, one can calculate the state of the memory register at any time t from

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Questions?