Bell's Theorem

Dr. Sung Lee

School of Mathematics and Natural Sciences, University of Southern Mississippi

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Outline

Pauli Spin Matrices

- The matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are called the *Pauli spin matrices*.

Pauli Spin Matrices

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- For any 3-dim unit vector $\vec{v}$, define

$$\vec{v} \cdot \vec{\sigma} = v_x\sigma_x + v_y\sigma_y + v_z\sigma z$$

Then $\vec{v} \cdot \vec{\sigma} = \begin{pmatrix} v_z & v_1 - iv_2 \\ v_1 + iv_2 & -v_z \end{pmatrix}$ is an observable and it has eigenvalues ± 1. Measurement of this observable is called a *measurement of spin along the* $\vec{v}$ *axis*.

Tensor Product

- Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then the *tensor product* (also called the *Kronecker product*) is defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

Tensor Product

- *Ex.* Let $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ |01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$|10\rangle = |1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ |11\rangle = |1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Tensor Product

- *Ex.* Let $A_0 = \sigma_z$ and $B_0 = -\frac{\sigma_x+\sigma_z}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$. Then

$$A_0 \otimes B_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Bell Inequality

- Suppose that Alice and Bob are in different locations far away from each other. Tom prepares a pair of particles and send one each to Alice and Bob. Alice performs one of the two possible measurements, say A_0 and A_1, associated with physical properties P_{A_0} and P_{A_1} of the particle she received. Each of A_0 and A_1 has 1 or -1 for the outcomes of measurement. When Bob receives one of the particles, he also performs one of the two possible measurements B_0 and B_1, each of which has outcome 1 or -1.

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- Let us consider $A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 = (A_0 + A_1)B_0 + (A_0 - A_1)B_1$. Since $A_0 = \pm 1$ and $A_1 = \pm 1$, either one of $A_0 + A_1$ and $A_0 - A_1$ is zero and the other is ± 2.

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- Let us consider $A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 = (A_0 + A_1)B_0 + (A_0 - A_1)B_1$. Since $A_0 = \pm 1$ and $A_1 = \pm 1$, either one of $A_0 + A_1$ and $A_0 - A_1$ is zero and the other is ± 2.
- *Bell inequality*: $\langle A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 \rangle \leq 2$

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1. The physical properties $P_{A_0}, P_{A_1}, P_{B_0}, P_{B_1}$ have definite values A_0, A_1, B_0, B_1 which exist independently of observation or measurement. This is called the *assumption of realism*.
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These two assumptions together are known as the *assumption of local realism*.

Quantum Measurement

- Tom prepares a quantum system of two qubits in the state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

This entangled quantum state of two qubits (two-qubit state) is called a *Bell state*, an *EPR state*, or an *EPR pair*.

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- Bob measures eaither one of the observables

$$B_0 = -\frac{\sigma_x + \sigma_z}{\sqrt{2}}, \ B_1 = \frac{\sigma_x - \sigma_z}{\sqrt{2}}$$

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- Since the system is in the state $|\psi\rangle$, the average value of $A_0 \otimes B_0$ is

$$\langle A_0 \otimes B_0 \rangle = \langle \psi | A_0 \otimes B_0 | \psi \rangle = \frac{1}{\sqrt{2}}$$

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- Similarly, the average values of the other observables are given by

$$\langle A_0 \otimes B_1 \rangle = \frac{1}{\sqrt{2}}, \ \langle A_1 \otimes B_0 \rangle = \frac{1}{\sqrt{2}}$$
$$\langle A_1 \otimes B_1 \rangle = \frac{1}{\sqrt{2}}$$

Quantum Measurement

- By the linearity of expected values, we obtain

$$\langle A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 \rangle = 2\sqrt{2} \geq 2$$

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- What this means is that

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- The violation of Bell inequality has been confirmed by experiments using photons. That is, Mother Nature does not obey Bell inequality.
- This implies that one or both of the two assumptions for the Bell inequality must be incorrect.
- Physicists have no consensus as to which of the two assumptions needs to be dropped.
- *Lesson*: Our common sense intuition about how the world works is not reliable.

Quantum Entanglement and Quantum Nonlocality

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- Consider an entangled pair of qubits belonging to Alice and Bob:
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- Suppose Alice and Bob are along way from each other.

Quantum Entanglement and Quantum Nonlocality

- Let $|a\rangle$ and $|b\rangle$ be the eigenstates of $\vec{v} \cdot \vec{\sigma}$. Then one can easily show that
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up to an unobservable global phase factor.
- Alice performs a measurement of spin along the $\vec{v}$ axis on her qubit, i.e. she measures the observable $\vec{v}\cdot\vec{\sigma}$ on her qubit. If Alice receives the result 1 (-1), she can predict with certainty Bob will receive the result -1 (1) when he measures spin along the $\vec{v}$ axis on his qubit.

La Fin

"*Quantum phenomena do not occur in a Hilbert space, they occur in a laboratory*."
Asher Peres