Surfaces of Revolution with Constant Mean Curvature in Hyperbolic 3-Space

Sungwook Lee
Department of Mathematics,
University of Southern Mississippi,
Hattiesburg, MS 39401, USA
sunglee@usm.edu
Kinsey-Ann Zarske
Department of Mathematics,
University of Southern Mississippi,
Hattiesburg, MS 39401, USA
kinseyann.zarske@eagles.usm.edu

January 19, 2014

Abstract

In this paper, we construct surfaces of revolution with constant mean curvature $H = c$ and minimal surfaces of revolution in hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of constant sectional curvature $-c^2$. It is shown that surfaces of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ tend toward the catenoid, the minimal surface of revolution in Euclidean 3-space $\mathbb{E}^3$ as $c \to 0$. Minimal surfaces of revolution in $\mathbb{H}^3(-c^2)$ also tend toward the catenoid in $\mathbb{E}^3$ as $c \to 0$.

2000 Mathematics Subject Classification: 53A10, 53C42, 53C50
Key words: constant mean curvature, flat chart model, minimal surface, surface of revolution, hyperbolic-3-space

Introduction

Surfaces of constant mean curvature $H = c$ in hyperbolic space $\mathbb{H}^3(-c^2)$ of constant sectional curvature $-c^2$ share many geometric properties in common with minimal surfaces in Euclidean 3-space $\mathbb{E}^3$ ([3]), although they live in two different spaces. It is not a coincidence. It turns out that there is a one-to-one correspondence, called the *Lawson correspondence*, between surfaces of constant mean curvature H_h in $\mathbb{H}^3(-c^2)$ and surfaces of constant mean curvature $H_e = \pm\sqrt{H_h^2 - c^2}$ in $\mathbb{E}^3$ ([6]). In particular, there is a one-to-one correspondence between surfaces of constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$

and minimal surfaces (i.e. surfaces of constant mean curvature $H = 0$) in $\mathbb{E}^3$. These corresponding constant mean curvature surfaces satisfy the same Gauss-Codazzi equations. $\mathbb{H}^3(-c^3)$ has a rotational symmetry, in fact SO(2) symmetry as its maximum rotational symmetry. So we may consider surfaces of revolution, in particular with constant mean curvature $H = c$. Surfaces of constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ can be in general constructed by Bryant's representation formula ([3]) using a holomorphic and a meromorphic functions analogously to the Weierstrass representation formula for minimal surfaces in $\mathbb{E}^3$ ([2], [9]), however it is not suitable to use to construct surfaces of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$.

In section 1, we introduce the *flat chart model* of $\mathbb{H}^3(-c^2)$. The flat chart model is convenient in many respects for our study of surfaces of revolution in $\mathbb{H}^3(-c^2)$. In section 2, we calculate the mean curvature of a parametric surface in $\mathbb{H}^3(-c^2)$. In section 4, we use this mean curvature formula to obtain the differential equation of the profile curve for a surface of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$. The differential equation is nonlinear and it cannot be solved analytically. By solving this equation numerically, we construct a surface of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$. In [10], Umehara and Yamada have shown that a minimal surface in $\mathbb{E}^3$ is the limit of surfaces of constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ as $c \to 0$ using the deformation of Lie groups. In section 4, it is shown that surfaces of revolution with constant mean curvature $H = c$ tend toward the catenoid, the minimal surfaces of revolution in $\mathbb{E}^3$ as $c \to 0$ in a trivial manner from the differential equation. In section 5, we illustrate the limiting behavior with graphics.

It turns out that minimal surfaces in $\mathbb{H}^3(-c^2)$ are not characterized by mean curvature unlike minimal surfaces in $\mathbb{E}^3$ or more generally in $\mathbb{E}^n$. In section 6, we construct minimal surfaces of revolution in $\mathbb{H}^3(-c^2)$ using the calculus of variations. The minimal surfaces of revolution in $\mathbb{H}^3(-c^2)$ also tend toward the catenoid in $\mathbb{E}^3$ as $c \to 0$.

The second named author Kinsey Zarske is an undergraduate student majoring mathematics and physics at the University of Southern Mississippi. A part of research presented in this paper was done as her undergraduate research project under the direction of the first named author.

1 The Flat Chart Model of Hyperbolic 3-Space $\mathbb{H}^3(-c^2)$

Let $\mathbb{R}^{3+1}$ denote the Minkowski spacetime with rectangular coordinates x^0, x^1, x^2, x^3 and the Lorentzian metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{1}$$

Hyperbolic 3-space is the hyperquadric

$$\mathbb{H}^3(-c^2) := \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^{3+1} : -(x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2 = -\frac{1}{c^2}\} \tag{2}$$

which has the constant sectional curvature $-c^2$. This is a hyperboloid of two sheets in spacetime so it is called the *hyperboloid model*[1] of hyperbolic 3-space. Consider the chart

$$U = \{(x^0, x^1, x^2, x^3) \in \mathbb{H}^3(-c^2) : x^0 + x^1 > 0\}$$

and define

$$\begin{aligned} t &= -\frac{1}{c}\log c(x^0 + x^1), \\ x &= \frac{x^2}{c(x^0 + x^1)}, \\ y &= \frac{x^3}{c(x^0 + x^1)}. \end{aligned} \tag{3}$$

Then

$$ds^2 = (dt)^2 + e^{-2ct}\{(dx)^2 + (dy)^2\}.$$

$\mathbb{R}^3$ with coordinates t, x, y and the metric

$$g_c := (dt)^2 + e^{-2ct}\{(dx)^2 + (dy)^2\} \tag{4}$$

is called the *flat chart model* of hyperbolic 3-space. We will still denote it by $\mathbb{H}^3(-c^2)$. The flat chart model is a local chart of hyperbolic 3-space, so it is not regarded as a standard model[2] of hyperbolic 3-space. As $c \to 0$, $\mathbb{H}^3(-c^2)$ flattens out to Euclidean 3-space $\mathbb{E}^3$. An interesting aspect of the flat chart model is that $(\mathbb{R}^3, g_c)$ is isometric to a Lie group G_c with a left-invariant metric [5]:

$$G_c = \left\{ \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & e^{ct} & 0 & x \\ 0 & 0 & e^{ct} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : (t, x, y) \in \mathbb{R}^3 \right\}.$$

In [5], M. Kokubu studied Weierstrass representation of minimal surfaces in $\mathbb{H}^3(-c^2)$ using the Lie group G_c and its Lie algebra.

2 Parametric Surfaces in $\mathbb{H}^3(-c^2)$

Let M be a domain[3] and $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ an immersion. The metric (4) induces an inner product $\langle\ ,\ \rangle$ on each tangent space $T_p\mathbb{H}^3(-c^2)$. Using this inner product, we can speak of conformal surfaces in $\mathbb{H}^3(-c^2)$.

Definition 1. *$\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ is said to be* conformal *if*

$$\begin{aligned} \langle \varphi_u, \varphi_v \rangle &= 0, \\ |\varphi_u| = |\varphi_v| &= e^{\omega/2}, \end{aligned} \tag{5}$$

[1] Often the connected component of the hyperboloid with $x^0 > 0$ is defined to be hyperbolic 3-space.

[2] As well-known there are 5 standard models of hyperbolic space [4].

[3] A 2-dimensional connected open set.

where (u, v) is a local coordinate system in M and $\omega : M \longrightarrow \mathbb{R}$ is a real-valued function in M.

The induced metric on the immersion is given by

$$ds_\varphi^2 = \langle d\varphi, d\varphi \rangle = e^\omega\{(du)^2 + (dv)^2\}. \tag{6}$$

A cross product can be defined locally on each tangent space $T_p\mathbb{H}^3(-c^2)$. Any $\mathbf{v}, \mathbf{w} \in T_p\mathbb{H}^3(-c^2)$ may be written as

$$\mathbf{v} = v_1\left(\frac{\partial}{\partial t}\right)_p + v_2\left(\frac{\partial}{\partial x}\right)_p + v_3\left(\frac{\partial}{\partial y}\right)_p,$$
$$\mathbf{w} = w_1\left(\frac{\partial}{\partial t}\right)_p + w_2\left(\frac{\partial}{\partial x}\right)_p + w_3\left(\frac{\partial}{\partial y}\right)_p,$$

where $\left\{\left(\frac{\partial}{\partial t}\right)_p, \left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p\right\}$ denote the canonical basis for $T_p\mathbb{H}^3(-c^2)$. Then the cross product $\mathbf{v} \times \mathbf{w}$ is defined to be

$$\begin{aligned}\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2)\left(\frac{\partial}{\partial t}\right)_p &+ e^{2ct}(v_3w_1 - v_1w_3)\left(\frac{\partial}{\partial x}\right)_p \\ &+ e^{2ct}(v_1w_2 - v_2w_1)\left(\frac{\partial}{\partial y}\right)_p\end{aligned} \tag{7}$$

where $p = (t, x, y) \in \mathbb{H}^3(-c^2)$. We can also write (7) simply as a determinant

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \frac{\partial}{\partial t} & e^{2ct}\frac{\partial}{\partial x} & e^{2ct}\frac{\partial}{\partial y} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{8}$$

One may also define a triple scalar product $\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle$ as a determinant

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle = \begin{vmatrix} u_1 & e^{2ct}u_2 & e^{2ct}u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{9}$$

However, as one can clearly see the cross product and the inner product is not interchangeable i.e.

$$\langle \mathbf{u}, \mathbf{v} \times \mathbf{w} \rangle \neq \langle \mathbf{u} \times \mathbf{v}, \mathbf{w} \rangle$$

unlike Euclidean case.

Let

$$E := \langle \varphi_u, \varphi_u \rangle, \ F := \langle \varphi_u, \varphi_v \rangle, \ G := \langle \varphi_v, \varphi_v \rangle. \tag{10}$$

Proposition 1. *Let $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ be an immersion. Then on each tangent plane $T_p\varphi(M)$,*

$$||\varphi_u \times \varphi_v||^2 = e^{4ct(u,v)}(EG - F^2) \tag{11}$$

where $p = (t(u, v), x(u, v), y(u, v)) \in \mathbb{H}^3(-c^2)$.

Proof. Straightforward by a direct calculation. □

Remark 1. If $c \to 0$, (11) becomes the familiar formula in Euclidean case [8]

$$||\varphi_u \times \varphi_v||^2 = EG - F^2.$$

3 The Mean curvature of a Parametric Surface in $\mathbb{H}^3(-c^2)$

In Euclidean case, the mean curvature of a parametric surface $\varphi(u, v)$ may be calculated by the Gauss' beautiful formula [8]

$$H = \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)} \tag{12}$$

where

$$\ell = \langle \varphi_{uu}, N \rangle, \ \mathfrak{m} = \langle \varphi_{uv}, N \rangle, \ \mathfrak{n} = \langle \varphi_{vv}, N \rangle$$

and N is the unit normal vector field of $\varphi(u, v)$. The proof of (12) in [8] is no longer valid for parametric surfaces in $\mathbb{H}^3(-c^2)$ since the Lagrange's identity does not hold for tangent vectors in $T_p\varphi(M)$. However, (12) is indeed valid for parametric surfaces in any 3-dimensional space including $\mathbb{H}^3(-c^2)$. For the proof, see appendix A.

Let $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ be a conformal surface satisfying (5) and N a unit normal vector field of φ. Let $S_p : T_p\varphi(M) \longrightarrow T_p\varphi(M)$ be the shape operator given by $S_p(\mathbf{v}) = -\nabla_{\mathbf{v}} N$ for $\mathbf{v} \in T_p\varphi(M)$. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix associated with shape operator with respect to the orthogonal basis φ_u, φ_v of $T_p\varphi(M)$. Then

$$\begin{aligned} S(\varphi_u) &= a\varphi_u + b\varphi_v, \\ S(\varphi_v) &= c\varphi_u + d\varphi_v. \end{aligned}$$

So,

$$\begin{aligned} \langle S(\varphi_u), \varphi_u \rangle + \langle S(\varphi_v), \varphi_v \rangle &= e^{\omega}(a + b) \\ &= e^{\omega}\mathrm{Tr}S \\ &= 2e^{\omega}H. \end{aligned}$$

On the other hand, for a fixed v_0, $\varphi(u, v_0)$ is a curve on the surface and consider N to be restricted on this curve. Then $S(\varphi_u) = -N_u$. Differentiating $\langle \varphi_u, N \rangle = 0$ with respect to u, we obtain

$$\begin{aligned} \langle \varphi_{uu}, N \rangle &= -\langle \varphi_u, N_u \rangle \\ &= \langle \varphi_u, S(\varphi_u) \rangle. \end{aligned}$$

Similarly, we also obtain $\langle S(\varphi_v), \varphi_v\rangle = \langle \varphi_{vv}, N\rangle$. Finally the mean curvature H is given by

$$\begin{aligned} H &= \frac{1}{2}e^{-\omega}(\langle \varphi_{uu}, N\rangle + \langle \varphi_{vv}, N\rangle) \\ &= \frac{1}{2}e^{-\omega}\langle \triangle\varphi, N\rangle \end{aligned}$$

where $\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Proposition 2. *Let $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ be a conformal surface satisfying* (5). *Then the mean curvature H of φ is computed to be*

$$H = \frac{1}{2}e^{-\omega}\langle \triangle\varphi, N\rangle. \tag{13}$$

One can easily see that the formulas (12) and (13) coincide for conformal surfaces.

4 Surfaces of Revolution with Constant Mean Curvature $H = c$ in $\mathbb{H}^3(-c^2)$

There is an interesting one-to-one correspondence between constant mean curvature surfaces in different Riemannian[4] space forms [6]. The Lawson correspondence is more than just a bijection. Those corresponding constant mean curvature surfaces satisfy the same Gauss-Codazzi equations, so they share many geometric properties in common, even though they live in different spaces. For this reason they are often called cousins. There is a one-to-one correspondence between surfaces of constant mean curvature H_h in $\mathbb{H}^3(-c^2)$ and surfaces of constant mean curvature[5]

$$H_e = \pm\sqrt{H_h^2 - c^2}. \tag{14}$$

In particular, surfaces of constant mean curvature $H = \pm c$ in $\mathbb{H}^3(-c^2)$ are cousins of minimal surfaces[6] in Euclidean 3-space. (14) tells that there is no surface in hyperbolic 3-space $\mathbb{H}^3(-c^2)$ with $H_h = 0$ unless $c = 0$ in which case the space is Euclidean 3-space $\mathbb{E}^3$. Note however that this does not mean there are no minimal surfaces in $\mathbb{H}^3(-c^2)$ in case readers are only familiar with minimal surfaces in Euclidean space. A parametric surface is called a *harmonic map* if it is a critical point of the area functional or the tension energy functional. A harmonic map is called a minimal surface if it is conformal. Physically, a minimal surface is an area minimizing or a surface tension energy minimizing surface. From (13), we see that $H = 0$ if and only if $\triangle\varphi = 0$. In Euclidean

[4]There is also Lawson correspondence between surfaces of constant mean curvature in different semi-Riemannian space forms. See [7] for details.

[5]The choice of $\pm$ signs depends on the orientation of the surface.

[6]Area minimizing surfaces or equivalently conformal surfaces with zero mean curvature.

3-space, φ is harmonic if and only if $\triangle\varphi = 0$, so a conformal parametric surface in $\mathbb{E}^3$ is minimal if and only if $H = 0$. (See [2], [9] for more details about minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^n$.) However, this is no longer true in $\mathbb{H}^3(-c^2)$ because the Laplace equation $\triangle\varphi = 0$ is not the harmonic map equation in $\mathbb{H}^3(-c^2)$ as shown in [5]. Minimal surfaces in $\mathbb{H}^3(-c^3)$ can be constructed in general using the Weierstrass representation formula obtained by M. Kokubu in [5]. In Section 6, we study how to construct minimal surface of revolution in $\mathbb{H}^3(-c^2)$ using the calculus of variations.

In this section, we are interested in constructing a surface of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ which corresponds to a minimal surface in $\mathbb{E}^3$ under the Lawson correspondence. It should be remarked that surfaces of constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ can be constructed in general with a holomorphic and a meromorphic data using Bryant's representation formula, analogously to Weierstrass representation formula for minimal surfaces in $\mathbb{E}^3$ [3].

From the metric (4), one can see that $\mathbb{H}^3(-c^2)$ has SO(2) symmetry i.e. SO(2) is a subgroup of the isometry group of $\mathbb{H}^3(-c^2)$ and it is the maximum rotational symmetry. More specifically, the rotations about the t-axis (i.e. rotations on the xy-plane) are the only type of Euclidean rotations that can be considered in $\mathbb{H}^3(-c^2)$.

Consider a profile curve $\alpha(u) = (g(u), h(u), 0)$ in the tx-plane. Denote by $\varphi(u, v)$ the rotation of $\alpha(u)$ about t-axis through an angle v. Then

$$\varphi(u, v) = (g(u), h(u)\cos v, h(u)\sin v). \tag{15}$$

If $g'(u)$ is never 0, (15) has a parametrization of the form

$$\varphi(w, v) = (w, f(w)\cos v, f(w)\sin v).$$

Thus, without loss of generality we may assume that $g(u) = u$ in (15). The quantities E, F, G are calculated to be

$$\begin{aligned}
E &= e^{-2cu}\{e^{2cu} + (h'(u))^2\},\\
F &= 0,\\
G &= e^{-2cu}(h(u))^2.
\end{aligned}$$

If we require $\varphi(u, v)$ to be conformal, then

$$e^{2cu} + (h'(u))^2 = (h(u))^2. \tag{16}$$

The quantities $\ell, \mathfrak{m}, \mathfrak{n}$ are calculated to be

$$\begin{aligned}
\ell &= -\frac{h''(u)h(u)}{\sqrt{(h(u))^2(e^{2cu} + (h'(u))^2)}},\\
\mathfrak{m} &= 0,\\
\mathfrak{n} &= \frac{(h(u))^2}{\sqrt{(h(u))^2(e^{2cu} + (h'(u))^2)}}.
\end{aligned}$$

So the mean curvature H is calculated by

$$\begin{aligned} H &= \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)} \\ &= \frac{1}{2}\frac{-h(u)h''(u) + e^{2cu} + (h'(u))^2}{e^{-2cu}(e^{2cu} + (h'(u))^2)\sqrt{(h(u))^2(e^{2cu} + (h'(u))^2)}}. \end{aligned}$$

With the conformality condition (16), H is reduced to

$$H = \frac{-h''(u) + h(u)}{2e^{-2cu}(h(u))^3}. \tag{17}$$

Let $H = c$. Then (17) can be written as

$$h''(u) - h(u) + 2ce^{-2cu}(h(u))^3 = 0. \tag{18}$$

Hence, constructing a surface of revolution with $H = c$ comes down to solving the second order nonlinear differential equation (18). The differential equation (18) cannot be solved analytically, so we solve it numerically with the aid of MAPLE software. (See appendix B for details of the computational procedure.) In the next section, we show the graphics of the surface of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ that we obtained using the numerical solution of the differential equation (18). The conformality condition (16) may be used to determine initial conditions. For all the numerical solutions of (18) in this paper, we used the same initial conditions $h(0) = 1.5$ and $h'(0) = 1.118$.

If $c \to 0$, then (18) becomes

$$h''(u) - h(u) = 0 \tag{19}$$

which is an equation of overdamped simple harmonic oscillator. (19) has the general solution

$$h(u) = c_1 \cosh u + c_2 \sinh u.$$

For $c_1 = 1, c_2 = 0$, $\varphi(u, v)$ is given by

$$\varphi(u, v) = (u, \cosh u \cos v, \cosh u \sin v). \tag{20}$$

This is a minimal surface of revolution in $\mathbb{E}^3$ which is called a *catenoid* since it is obtained by rotating a catenary $h(u) = \cosh u$. Figure 1 shows the catenoid (20).

5 The Illustration of the Limit of Surfaces of Revolution with $H = c$ in $\mathbb{H}^3(-c^2)$ as $c \to 0$

In section 4, it is shown that the limit of surfaces of revolution with constant mean curvature $H = c$ in $\mathbb{H}^3(-c^2)$ is the catenoid, the minimal surface of revolution in $\mathbb{E}^3$. In this section, such limiting behavior of surfaces of revolution

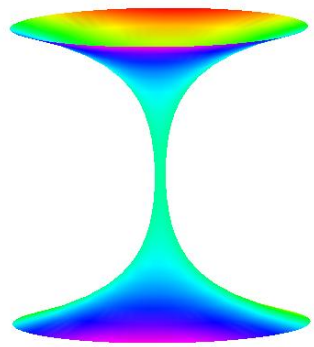
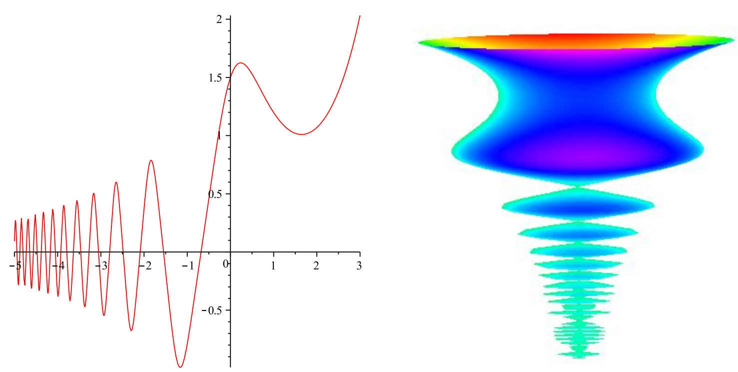


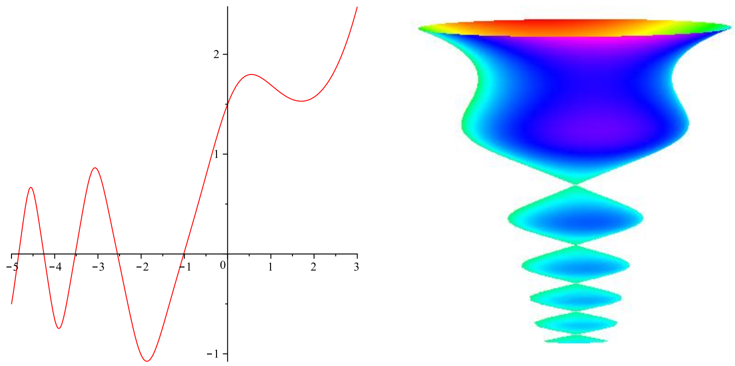
Fig. 1: Catenoid in $\mathbb{E}^3$



(a) (b)

Fig. 2: CMC $H = 1$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 3$, (b) Surface of Revolution in $\mathbb{H}^3(-1)$

with $H = c$ in $\mathbb{H}^3(-c^2)$ is illustrated with graphics in Figure 2 ($H = 1$), Figure 3 ($H = \frac{1}{2}$), Figure 4 ($H = \frac{1}{4}$), Figure 5 ($H = \frac{1}{8}$), Figure 6 ($H = \frac{1}{16}$), Figure 7 ($H = \frac{1}{64}$), and Figure 8 ($H = \frac{1}{256}$). Figure 8 (b) already looks pretty close to the catenoid in Figure 1. In order to visualize better the limiting behavior of surfaces of revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$ as $c \to 0$, the first named author has made some animations available in his website. An animation of profile curves for CMC $H = c$ surfaces of revolution in $\mathbb{H}^3(-c^2)$ tending toward the profile curve of the catenoid in $\mathbb{E}^3$ as $c \to 0$ is available at `http://www.math.usm.edu/lee/profileanim.gif`. An animation of CMC $H = c$ surfaces of revolution in $\mathbb{H}^3(-c^2)$ tending toward the catenoid in $\mathbb{E}^3$ as $c \to 0$ is available at `http://www.math.usm.edu/lee/cmcanim.gif`. The same animation of CMC $H = c$ surfaces of revolution in $\mathbb{H}^3(-c^2)$ with the catenoid in $\mathbb{E}^3$ is available at `http://www.math.usm.edu/lee/cmcanim2.gif`.



(a) (b)

Fig. 3: CMC $H = \frac{1}{2}$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 3$, (b) Surface of Revolution in $\mathbb{H}^3\left(-\frac{1}{4}\right)$

6 Minimal Surface of Revolution in $\mathbb{H}^3(-c^2)$

In section 4, we pointed out that minimal surfaces in $\mathbb{H}^3(-c^2)$ are no longer characterized by mean curvature. In this section, we find the minimal surface of revolution in $\mathbb{H}^3(-c^2)$ as a critical point of the area functional using the calculus of variations.

Let us consider a surface of revolution which is obtained by revolving a curve $x(t)$ in the tx-plane about the t-axis. The curve is required to pass through the points (t_1, x_1) and (t_2, x_2) as seen in Figure 9. Our variational problem is to choose the curve $x(t)$ so that the area of the resulting surface of revolution is a minimum. The area element dA shown in Figure 9 is given by

$$dA = 2\pi x(t) ds = 2\pi x(t)\sqrt{1 + e^{-2ct}\dot{x}^2}dt, \tag{21}$$

where $\dot{x} = \frac{dx(t)}{dt}$. The area functional is then

$$J = \int_{t_1}^{t_2} 2\pi x(t)\sqrt{1 + e^{-2ct}\dot{x}^2}dt. \tag{22}$$

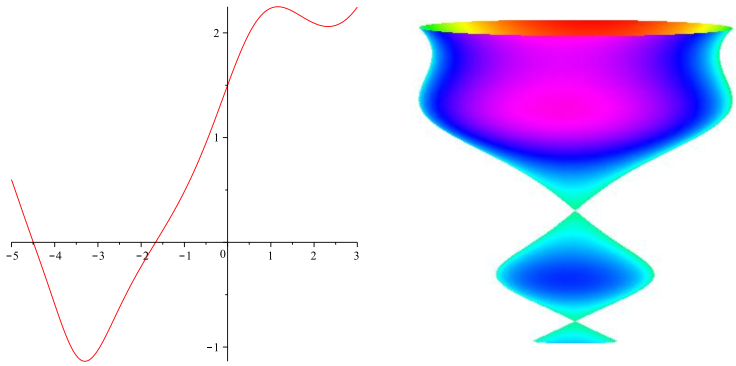
Let[7]

$$f(x, \dot{x}, t) = x\sqrt{1 + e^{-2ct}\dot{x}^2}.$$

Finding a critical point of the area functional (22) is equivalent to solving the Euler-Lagrange equation (see [1] for example)

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0 \tag{23}$$

[7] The constant 2π is neglected since it makes no contribution to the solution of our variational problem.



(a)

(b)

Fig. 4: CMC $H = \frac{1}{4}$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 3$, (b) Surface of Revolution in $\mathbb{H}^3\left(-\frac{1}{16}\right)$

which is equivalent to the second order nonlinear differential equation

$$1 + e^{-2ct}\dot{x}^2 + xce^{-4ct}\dot{x}^3 + 2xce^{-2ct}\dot{x} - xe^{-2ct}\ddot{x} = 0. \tag{24}$$

Recall that a minimal surface is a conformal harmonic map. Applying the conformality condition (16), the equation (24) simplifies to

$$\ddot{x} - c(1 + e^{-2ct}x^2)\dot{x} - x = 0. \tag{25}$$

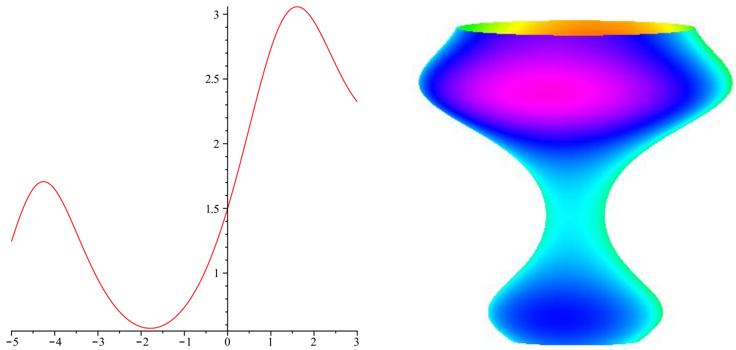
This nonlinear differential equation (25) cannot be solved analytically and again we need to solve it numerically. Figure 10 shows the profile curve $x(t)$ and the minimal surface of revolution in $\mathbb{H}^3(-1)$. For the numerical solution, we also used the same initial conditions $x(0) = 1.5$ and $\dot{x}(0) = 1.118$ as before.

If $c \to 0$, then (25) becomes the equation of overdamped simple harmonic oscillator (19). Hence, as $c \to 0$ minimal surfaces of revolution in $\mathbb{H}^3(-c^2)$ also tend toward the catenoid, the minimal surface of rotation in $\mathbb{E}^3$. An animation of this limiting behavior of minimal surfaces in $\mathbb{H}^3(-c^2)$ is available at `http://www.math.usm.edu/lee/minimal_animate.gif`.

A The Proof of the Gauss' Formula (12)

Let $\varphi : M(u, v) \longrightarrow \mathcal{M}^3$ be a parametric surface in a 3-dimensional differentiable manifold $\mathcal{M}^3$. Let $S : T\varphi(M) \longrightarrow T\varphi(M)$ be the shape operator given by $S(\mathbf{v}) = -\nabla_{\mathbf{v}} N$ for $\mathbf{v} \in T\varphi(M)$. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix associated with the shape operator with respect to the basis φ_u, φ_v of $T\varphi(M)$. Then

$$\begin{aligned} S(\varphi_u) &= a\varphi_u + b\varphi_v, \\ S(\varphi_v) &= c\varphi_u + d\varphi_v \end{aligned}$$



(a) (b)

Fig. 5: CMC $H = \frac{1}{8}$: (a) Profile Curve $h(u)$, $-5 \le u \le 3$, (b) Surface of Revolution in $\mathbb{H}^3\left(-\frac{1}{64}\right)$

and

$$\langle S(\varphi_u), \varphi_u\rangle = a\langle \varphi_u, \varphi_u\rangle + b\langle \varphi_v, \varphi_u\rangle, \tag{26}$$
$$\langle S(\varphi_u), \varphi_v\rangle = a\langle \varphi_u, \varphi_v\rangle + b\langle \varphi_v, \varphi_v\rangle, \tag{27}$$
$$\langle S(\varphi_v), \varphi_u\rangle = c\langle \varphi_u, \varphi_u\rangle + d\langle \varphi_v, \varphi_u\rangle, \tag{28}$$
$$\langle S(\varphi_v), \varphi_v\rangle = c\langle \varphi_u, \varphi_v\rangle + d\langle \varphi_v, \varphi_v\rangle. \tag{29}$$

From (26) and (27), we find

$$a = \frac{\langle S(\varphi_u), \varphi_u\rangle\langle \varphi_v, \varphi_v\rangle - \langle S(\varphi_u), \varphi_v\rangle\langle \varphi_v, \varphi_u\rangle}{\langle \varphi_u, \varphi_u\rangle\langle \varphi_v, \varphi_v\rangle - \langle \varphi_u, \varphi_v\rangle^2},$$

and from (28) and (29), we find

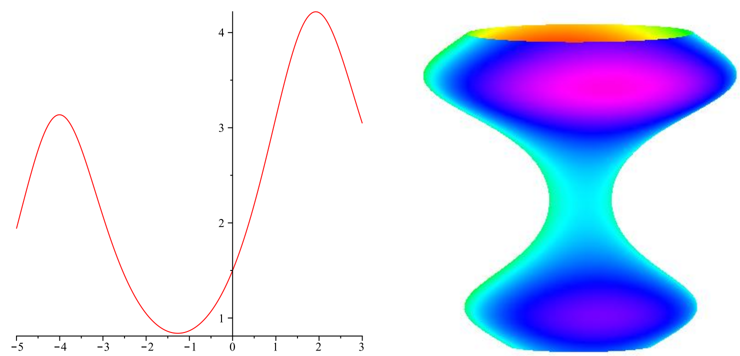
$$d = -\frac{\langle S(\varphi_v), \varphi_u\rangle\langle \varphi_u, \varphi_v\rangle - \langle S(\varphi_v), \varphi_v\rangle\langle \varphi_u, \varphi_u\rangle}{\langle \varphi_u, \varphi_u\rangle\langle \varphi_v, \varphi_v\rangle - \langle \varphi_u, \varphi_v\rangle^2}.$$

The mean curvature H is

$$\begin{aligned} H &= \frac{1}{2}\mathrm{tr}S \\ &= \frac{a+d}{2} \\ &= \frac{1}{2}\frac{\langle S(\varphi_u), \varphi_u\rangle\langle \varphi_v, \varphi_v\rangle + \langle S(\varphi_v), \varphi_u\rangle\langle \varphi_u, \varphi_v\rangle - 2\langle S(\varphi_u), \varphi_v\rangle\langle \varphi_v, \varphi_u\rangle}{\langle \varphi_u, \varphi_u\rangle\langle \varphi_v, \varphi_v\rangle - \langle \varphi_u, \varphi_v\rangle^2}. \end{aligned} \tag{30}$$

For a fixed v_0, $\varphi(u, v_0)$ is a curve on the surface and consider N to be restricted on this curve. Then $S(\varphi_u) = -N_u$. Similarly, $S(\varphi_v) = -N_v$. Differentiating $\langle \varphi_u, N\rangle = \langle \varphi_v, N\rangle = 0$ with respect to u and with respect to v, we obtain

$$\langle \varphi_u, N_v\rangle = -\langle \varphi_{uv}, N\rangle = \langle \varphi_v, N_u\rangle$$



(a) (b)

Fig. 6: CMC $H = \frac{1}{16}$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 3$, (b) Surface of Revolution in $\mathbb{H}^3\left(-\frac{1}{256}\right)$

i.e.

$$\langle \varphi_u, S(\varphi_v)\rangle = \langle \varphi_{uv}, N\rangle = \langle \varphi_v, S(\varphi_u)\rangle.$$

Differentiating $\langle \varphi_u, N\rangle = 0$ with respect to u, we obtain

$$\langle \varphi_u, S(\varphi_u)\rangle = \langle \varphi_{uu}, N\rangle.$$

Similarly, we also obtain

$$\langle \varphi_v, S(\varphi_v)\rangle = \langle \varphi_{vv}, N\rangle.$$

Therefore, (30) can be written as (12).

B The Numerical Solution of (18) with MAPLE

The numerical solution of the differential equation (18) was obtained with the aid of MAPLE software version 15. For the readers who want to try by themselves, here are the MAPLE commands that the authors used to obtain the numerical solutions and the graphics. The commands need to be run in the following order.

First we clear the memory.

```
restart:
```

In order to solve the equation numerically, we need a MAPLE package called DEtools.

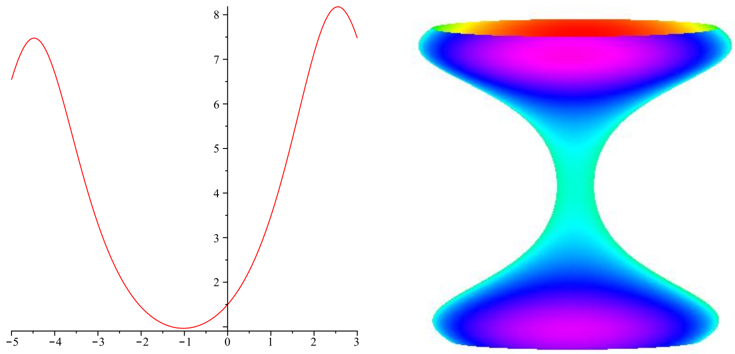
```
with(DEtools):
```

Set the c value. In this example, we set $c = 1$.

```
c:=1;
```

Define the differential equation (18).

```
eq:=diff(h(u),u,u)-h(u)+2*exp(-2*c*u)*c*h(u)^3=0;
```



(a) (b)

Fig. 7: CMC $H = \frac{1}{64}$: (a) Profile Curve $h(u)$, $-5 \leq u \leq 3$, (b) Surface of Revolution in $\mathbb{H}^3\left(-\frac{1}{4096}\right)$

Define the initial conditions for the equation (18).

```
ic:=h(0)=1.5,D(h)(0)=1.118;
```

Get the numerical solution.

```
sol:=dsolve({eq,ic},numeric,output=listprocedure);
```

Define the numerical solution as a function Y.

```
Y:=subs(sol,h(u)):
```

For testing, we evaluate $Y(.8)$.

```
Y(.8);
```

The output is

```
1.32418662912977
```

Now, we are ready to plot the profile curve $h(u)$.

```
plot(Y,-5..3,scaling=constrained);
```

The output is Figure 2 (a).

In order to plot surfaces, we need plot3d which is a part of the package called plots.

```
with(plots);
```

Define the surface of revolution X.

```
X:=[u,Y(u)*cos(v),Y(u)*sin(v)];
```

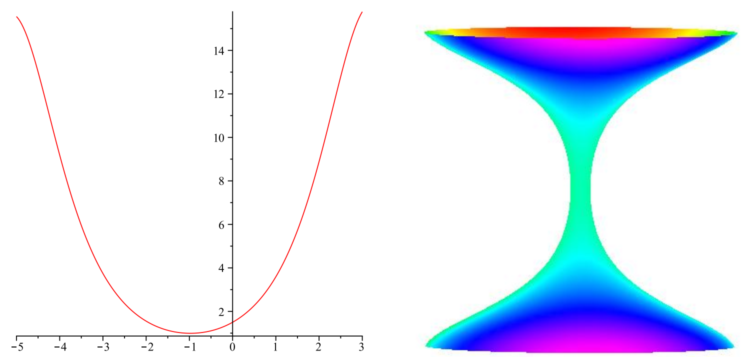
Finally, we plot the surface of revolution X.

```
plot3d(X,u=-5..3,v=0..2*Pi,grid=[85,85],style=patchnogrid,
shading=zhue,orientation=[62,64]);
```

The output is Figure 2 (b).

References

[1] G. Arfken, Mathematical Methods for Physicists, 3rd Edition, Academic Press, 1985



(a) (b)

Fig. 8: CMC $H = \frac{1}{256}$: (a) Profile Curve $h(u)$, $-5 \le u \le 3$, (b) Surface of Revolution in $\mathbb{H}^3\left(-\frac{1}{65535}\right)$

[2] J. Lucas M. Barbosa and A. Gervasio Colares, Minimal Surfaces in $\mathbb{R}^3$, Lecture Notes in Mathematics **1195**, Springer, 1986

[3] R. L. Bryant, *Surfaces of mean curvature one in hyperbolic space*, Astérisque **12**, No. 154-155, 321–347 (1988)

[4] J. W. Canon, W. J. Floyd, R. Kenyon, and W. R. Parry, *Hyperbolic Geometry*, Flavors of Geometry, MSRI Publications **31**, 59–115 (1997)

[5] M. Kokubu, *Weierstrass Representation for Minimal Surfaces in Hyperbolic Space*, Tohoku Math. J. **49**, 367–377 (1997)

[6] H. Blaine Lawson, Jr., *Complete minimal surfaces in* S^3, Ann. of Math. **92**, 335–374 (1970)

[7] S. Lee, *Timelike surfaces of constant mean curvature* ± 1 *in anti-de Sitter 3-space* $\mathbb{H}^3_1(-1)$, Ann. Global Anal. Geom. **29**, no. 4, 361–407 (2006)

[8] B. O'Neill, Elementary Differential Geometry, Academic Press, 1967

[9] R. Osserman, A Survey of Minimal Surfaces, Dover Publications, 1986

[10] M. Umehara and K. Yamada, *A parametrization of the Weierstrass formulae and perturbation of complete minimal surfaces in* $\mathbb{R}^3$ *into the hyperbolic 3-space*, J. Reine Angew. Math. **432**, 93–116 (1992)

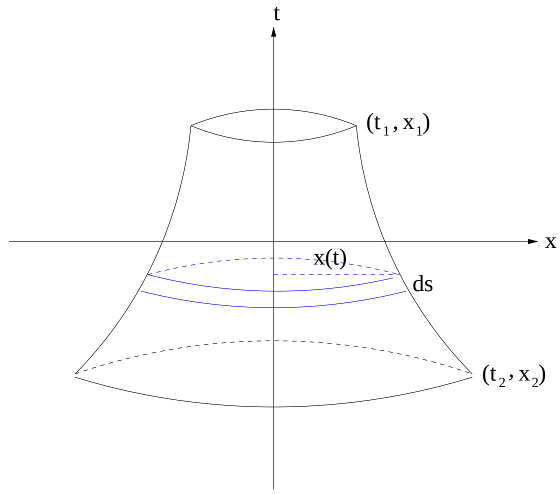
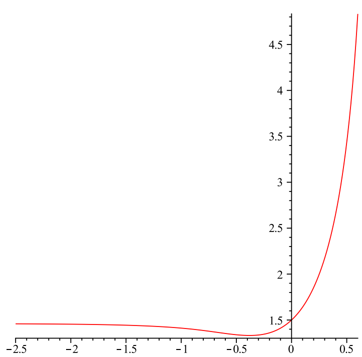
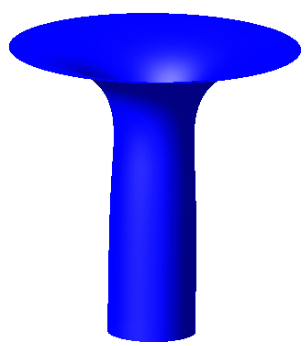


Fig. 9: Surface of Revolution in $\mathbb{H}^3(-c^2)$



(a)



(b)

Fig. 10: (a) Profile Curve $x(t)$, $-2.5 \le t \le 0.6$, (b) Minimal Surface of Revolution in $\mathbb{H}^3(-1)$