FLAT LORENTZ SURFACES IN ANTI-DE SITTER 3-SPACE AND GRAVITATIONAL INSTANTONS

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Abstract. In this paper, we study flat Lorentz surfaces (i.e. conformal timelike surfaces)in anti-de Sitter 3-space $\mathbb{H}^3_1(-1)$ in terms of the conformal structure determined by the second fundamental form (the second conformal structure). Those flat Lorentz surfaces can be represented in terms of a Lorentz holomorphic and a Lorentz anti-holomorphic data with respect to the second conformal structure. The conformality of the hyperbolic Gauß map is also discussed. Using the connection of flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$ to a hyperbolic Monge-Ampère equation, we find that there is a correspondence between flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$ and a class of anti-self-dual gravitational instantons.

Introduction

Let $\bar{M}$ be a semi-Riemannian manifold and $M \subset \bar{M}$ a hypersurface with the sectional curvatures $\bar{K}$ and K, respectively. Let S be the shape operator derived from the unit normal vector field N on the hypersurface M. If X, Y span a nondegenerate tangent plane on M, then the Gauß equation is given by (see for instance [14])

$$K(X,Y) = \bar{K}(X,Y) + \epsilon \frac{\langle S(X), X\rangle \langle S(Y), Y\rangle - \langle S(X), Y\rangle^2}{\langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2}, \tag{1}$$

where $\epsilon = \langle N, N\rangle$. If $\bar{M}$ is a 3-dimensional space form i.e. a 3-dimensional space of constant sectional curvature, say κ, and $M \subset \bar{M}$ a surface. Then (1) is written as

$$K = \kappa + \epsilon \frac{\det II}{\det I} \tag{2}$$

where K is the Gaußian curvature of M, and I, II denote the first and the second fundamental forms of M respectively. If $\bar{M}$ is flat i.e. $K = 0$, then we have

$$\det II = -\kappa\epsilon \det I. \tag{3}$$

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It follows from (3) that if $-\kappa\epsilon > 0$ then the second fundamental form II may determine a conformal structure on $\bar{M}$. Such a conformal structure is called the *second conformal structure*. In order for $\bar{M}$ to have the second conformal structure, either $\kappa > 0, \epsilon < 0$ or $\kappa < 0, \epsilon > 0$. Thus, if M is Riemannian, $M = \mathbb{H}^3(-c^2)$, hyperbolic 3-space of sectional curvature $-c^2$, is the only 3-dimensional space form in which flat surfaces can have the second conformal structure. If M is semi-Riemannian[1], only flat spacelike surfaces in $\mathbb{S}^3_1(c^2)$, de Sitter 3-space of sectional curvature c^2, and flat timelike surfaces in $\mathbb{H}^3_1(-c^2)$, anti-de Sitter 3-space of sectional curvature $-c^2$, can have the second conformal structure. Flat surfaces in $\mathbb{H}^3(-1)$ and flat spacelike surfaces in $\mathbb{S}^3_1(1)$ cases are studied by Gálvez, Martínez, and Milán in [6] and [7], respectively. In this paper, we study the only remaining case, flat timelike surfaces in $\mathbb{H}^3_1(-1)$. It turns out that flat timelike surfaces in $\mathbb{H}^3_1(-1)$ can be obtained by a representation formula in terms of a Lorentz holomorphic and a Lorentz anti-holomorphic data with respect to the second conformal structure. The conformality of the hyperbolic Gauß map of timelike surfaces in $\mathbb{H}^3_1(-1)$ is also studied. We show that the hyperbolic Gauß map of a timelike surface in $\mathbb{H}^3_1(-1)$ is conformal with respect to the second conformal structure if and only if the timelike surface is flat and totally umbilic. Flat timelike surfaces are associated with a hyperbolic Monge-Amère equation. Using this connection, we show that there is a correspondence between flat timelike surfaces in $\mathbb{H}^3_1(-1)$ and a class of anti-self-dual gravitational instantons.

1. Lorentz surfaces in anti-de Sitter 3-space

Let $\mathbb{E}^4_2$ be the semi-Euclidean 4-space with coordinates (x^0, x^1, x^2, x^3) and the semi-Riemannian metric $\langle\ ,\ \rangle$ of signature $(-,-,+,+)$ given by the quadratic form $-(dx^0)^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. The anti-de Sitter 3-space $\mathbb{H}^3_1(-1)$ is a Lorentzian 3-manifold of constant sectional curvature -1 that can be realized as the hyperquadric in $\mathbb{E}^4_2$:

$$\mathbb{H}^3_1(-1) := \{(x^0, x^1, x^2, x^3) \in \mathbb{E}^4_2 : -(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -1\}.$$

Let $\mathbb{D}$ be a 2-dimensional orientable domain[2] and $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ an immersion. The immersion φ is said to be *timelike* if the induced metric I on $\mathbb{D}$ is Lorentzian. The induced Lorentzian metric I determines a Lorentz conformal structure $\mathcal{C}_I$ on $\mathbb{D}$. More specifically, if (x', y') is a Lorentz isothermal coordinate system with respect to the conformal structure $\mathcal{C}_I$, then the first fundamental form is given by $I = e^\rho\{-(dx')^2+(dy')^2\}$ where ρ is a real-valued smooth function defined on $\mathbb{D}$. Hence a timelike immersion φ being conformal

[1] also called pseudo-Riemannian

[2] A connected open set as usual.

is equivalent to the conditions:

$$\begin{aligned} &\langle \varphi_{x'}, \varphi_{x'} \rangle = -e^{\rho},\ \langle \varphi_{y'}, \varphi_{y'} \rangle = e^{\rho}, \\ &\langle \varphi_{x'}, \varphi_{y'} \rangle = 0. \end{aligned} \tag{4}$$

These conditions are said to be *conformality conditions* and a conformal timelike surface is said to be a *Lorentz surface* hereafter. Let $u' := x' + y'$ and $v' := -x' + y'$. Then (u', v') defines a *null coordinate system* with respect to the conformal structure $\mathcal{C}_I$. The first fundamental form I is written in terms of (u', v') as

$$I = e^{\rho} du' dv'.$$

The differential operators $\frac{\partial}{\partial u'}$ and $\frac{\partial}{\partial v'}$ are given by

$$\frac{\partial}{\partial u'} = \frac{1}{2}\left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right),\ \frac{\partial}{\partial v'} = \frac{1}{2}\left(-\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right).$$

With these differential operators, one can speak of Lorentz holomorphicity and Lorentz anti-holomorphicity. A map $f : \mathbb{D} \longrightarrow \mathbb{E}^2_1$ is said to be *Lorentz holomorphic* (*Lorentz anti-holomorphic*) if $\frac{\partial f}{\partial v'} = 0$ ($\frac{\partial f}{\partial u'} = 0$, respectively).

The conformality conditions (4) are equivalent to

$$\langle \varphi_{u'}, \varphi_{u'} \rangle = \langle \varphi_{v'}, \varphi_{v'} \rangle = 0,\ \langle \varphi_{u'}, \varphi_{v'} \rangle = \frac{1}{2} e^{\rho}.$$

Let N be a unit normal vector field of $\mathbb{D}$. Then

$$\langle N, N \rangle = 1,\ \langle \varphi, N \rangle = \langle \varphi_{u'}, N \rangle = \langle \varphi_{v'}, N \rangle = 0.$$

The mean curvature H is computed to be $H = 2e^{-\rho}\langle \varphi_{u'v'}, N \rangle$. Let $Q := \langle \varphi_{u'u'}, N \rangle$ and $R := \langle \varphi_{v'v'}, N \rangle$. The quadratic differential

$$\mathcal{Q} := Q du'^2 + R dv'^2$$

is then called *Hopf differential*. The Hopf differential is defined globally on the Lorentz surface $(\mathbb{D}, \mathcal{C}_I)$. The second fundamental form II of $\mathbb{D}$ is given by

$$II = \mathcal{Q} + HI.$$

A point $p \in \mathbb{D}$ is said to be an *umbilic point* if II is proportional to I at p or equivalently p is a common zero of Q and R, i.e. $\mathcal{Q}(p) = 0$.

If K is the Gaußian curvature, then the Gauß equation which describes a relationship between K, H, Q, R takes the form :

$$H^2 - K - 1 = 4e^{-2\rho} QR. \tag{5}$$

The semi-Euclidean 4-space $\mathbb{E}^4_2$ is identified with the linear space $\mathrm{M}(2, \mathbb{R})$ of all 2×2 real matrices via the correspondence

$$\mathbf{u} = (x^0, x^1, x^2, x^3) \longleftrightarrow \begin{pmatrix} x^0 + x^3 & x^1 + x^2 \\ -x^1 + x^2 & x^0 - x^3 \end{pmatrix}. \tag{6}$$

The inner product of $\mathbb{E}^4_2$ corresponds to the inner product of $\mathrm{M}(2,\mathbb{R})$

$$\langle \mathbf{u},\mathbf{v}\rangle = \frac{1}{2}\{\mathrm{tr}(\mathbf{u}\mathbf{v}) - \mathrm{tr}(\mathbf{u})\,\mathrm{tr}(\mathbf{v})\},\ \ \mathbf{u},\mathbf{v}\in \mathrm{M}(2,\mathbb{R}). \tag{7}$$

In particular, $\langle \mathbf{u},\mathbf{u}\rangle = -\det \mathbf{u}$ so the correspondence is an isometry. The standard basis $\{e_0,e_1,e_2,e_3\}$ for $\mathbb{E}^4_2$ is then identified with the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \mathbf{j}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathbf{k}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\{\mathbf{1},\mathbf{i},\mathbf{j}',\mathbf{k}'\}$ satisfies the properties:

$$\begin{aligned} &\mathbf{i}^2 = -\mathbf{1},\ {\mathbf{j}'}^2 = {\mathbf{k}'}^2 = \mathbf{1}, \\ &\mathbf{i}\mathbf{j}' = -\mathbf{j}'\mathbf{i} = \mathbf{k}',\ \mathbf{j}'\mathbf{k}' = -\mathbf{k}'\mathbf{j}' = -\mathbf{i},\ \mathbf{k}'\mathbf{i} = -\mathbf{i}\mathbf{k}' = \mathbf{j}'. \end{aligned}$$

A 2×2 matrix of the form $x^0\mathbf{1} + x^1\mathbf{i} + x^2\mathbf{j}' + x^3\mathbf{k}'$ is called a *split-quaternion* or a *paraquaternion.* The set $\mathbb{H}'$ of all split-quaternions is an algebra over real numbers and by (6) $\mathbb{H}'$ is identified with $\mathbb{E}^4_2$. The group G of timelike unit vectors corresponds to a special linear group

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2,\mathbb{R}) : ad - bc = 1 \right\}.$$

The metric of G induced by the inner product (7) is a bi-invariant Lorentz metric of constant curvature -1. Hence, G is identified with $\mathbb{H}^3_1(-1)$.

The Lie group $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ acts isometrically on $\mathbb{E}^4_2$ via the group action:

$$(g_1,g_2)\cdot\mathbf{u} = g_1\mathbf{u}g_2^t \tag{8}$$

for $g_1,g_2\in\mathrm{SL}(2,\mathbb{R})$ and $\mathbf{u}\in\mathbb{E}^4_2$. This action is transitive on $\mathbb{H}^3_1(-1)$. The isotropy group of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ at $\mathbf{1}$ is $\mathcal{K} = \{(g,(g^{-1})^t) : g\in\mathrm{SL}(2,\mathbb{R})\}$ and $\mathbb{H}^3_1(-1)$ is represented as the Lorentzian symmetric space $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})/\mathcal{K}$. The natural projection $\pi : \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathbb{H}^3_1(-1)$ is given by $\pi(g_1,g_2) = g_1g_2^t$.

It is worth noting that there is another useful action of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{E}^4_2$, the so-called *diagonal action*:

$$(g_1,g_2)\cdot\mathbf{u} = g_1\mathbf{u}g_2^{-1} \tag{9}$$

for $g_1,g_2\in\mathrm{SL}(2,\mathbb{R})$ and $\mathbf{u}\in\mathbb{E}^4_2$. This action is again isometric on $\mathbb{E}^4_2$ and transitive on $\mathbb{H}^3_1(-1)$. The reason why it is called the diagonal action is that the isotropy subgroup of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ at $\mathbf{1}$ is the diagonal subgroup $\triangle = \{(g,g) : g\in\mathrm{SL}(2,\mathbb{R})\}$; hence $\mathbb{H}^3_1(-1)$ can be also represented as the Lorentzian symmetric space $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})/\triangle$. The natural projection $\pi : \mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathbb{H}^3_1(-1)$ is then given by $\pi(g_1,g_2) = g_1g_2^{-1}$. Hereafter we use the action (8) only but it should be mentioned that this action is not particularly more advantageous than the diagonal action (9).

The action (8) induces (so does the diagonal action) a double covering $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \longrightarrow \mathrm{SO}^{++}(2,2)$, where $\mathrm{SO}^{++}(2,2)$ denotes the identity component of the pseudo-orthogonal group $\mathrm{O}(2,2)$. The frame field $\{e_\alpha : \alpha = 0,1,2,3\}$ can be then parametrised as follows: for each $g = (g_1, g_2) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$,

$$e_0(g) = g_1 \mathbf{1} g_2^t,\ e_1(g) = g_1 \mathbf{i} g_2^t,\ e_2(g) = g_1 \mathbf{j}' g_2^t,\ e_3(g) = g_1 \mathbf{k}' g_2^t.$$

Let $\mathbb{D}$ be a 2-dimensional simply connected orientable domain and $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ a Lorentz surface with unit normal vector field N. Then we can define an orthonormal frame field $\mathcal{F} : \mathbb{D} \longrightarrow \mathrm{SO}^{++}(2,2)$ along φ by

$$\begin{aligned}\mathcal{F} &= (\varphi, e^{-\rho/2}\varphi_{x'}, e^{-\rho/2}\varphi_{y'}, N) \\ &= (\varphi, e^{-\rho/2}(\varphi_{u'} - \varphi_{v'}), e^{-\rho/2}(\varphi_{u'} + \varphi_{v'}), N).\end{aligned} \tag{10}$$

By means of a double covering induced by the group action (8), one can find a lift $F = (F_1, F_2)$ (called a coordinate frame) of $\mathcal{F}$ to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ such that

$$F_1(\mathbf{1}, \mathbf{i}, \mathbf{j}', \mathbf{k}') F_2^t = \mathcal{F}. \tag{11}$$

Each component framing F_1 and F_2 satisfy the following system of first order linear equations, so-called *Lax system*:

$$\begin{aligned}(F_1)_{u'} = F_1 U_1,\ (F_1)_{v'} = F_1 V_1, \\ (F_2)_{u'} = F_2 U_2,\ (F_2)_{v'} = F_2 V_2\end{aligned} \tag{12}$$

where

$$\begin{aligned} U_1 &= \begin{pmatrix} \rho_{u'}/4 & \frac{1}{2}e^{\rho/2}(H+1) \\ -e^{-\rho/2}Q & -\rho_{u'}/4 \end{pmatrix},\ V_1 = \begin{pmatrix} -\rho_{v'}/4 & e^{-\rho/2}R \\ -\frac{1}{2}e^{\rho/2}(H-1) & \rho_{v'}/4 \end{pmatrix}, \\ U_2 &= \begin{pmatrix} -\rho_{u'}/4 & e^{-\rho/2}Q \\ -\frac{1}{2}e^{\rho/2}(H-1) & \rho_{u'}/4 \end{pmatrix},\ V_2 = \begin{pmatrix} \rho_{v'}/4 & \frac{1}{2}e^{\rho/2}(H+1) \\ -e^{-\rho/2}R & -\rho_{v'}/4 \end{pmatrix}. \end{aligned}$$

The compatibility condition $F_{u'v'} = F_{v'u'}$ gives the Maurer-Cartan equations

$$(U_1)_{v'} - (V_1)_{u'} - [U_1, V_1] = 0, \tag{13}$$

$$(U_2)_{v'} - (V_2)_{u'} - [U_2, V_2] = 0. \tag{14}$$

Each of these equations is equivalent to the Gauß-Mainardi-Codazzi Equations

$$\rho_{u'v'} + \frac{1}{2}e^{\rho}(H^2 - 1) - 2QRe^{-\rho} = 0, \tag{15}$$

$$H_{u'} = 2e^{-\rho}Q_{v'},\ H_{v'} = 2e^{-\rho}R_{u'}. \tag{16}$$

2. Fundamental Equations

In this section we derive some fundamental equations that we need in order to study flat Lorentz surfaces in the following sections.

Proposition 1. *Let $\mathbb{D}$ be a 2-dimensional simply connected domain with isothermal coordinates (x', y') and $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ a flat Lorentz surface with the first fundamental form $I = e^\rho(-dx'^2 + dy'^2)$. Then there exist coordinates (x, y) in $\mathbb{D}$ so that I can be written as*

$$I = -dx^2 + dy^2. \tag{17}$$

Proof. From equations (5) and (15), we obtain

$$\rho_{u'v'} = -\frac{1}{2}e^\rho K. \tag{18}$$

Since $K = 0$, (18) is simply the homogeneous wave equation

$$\rho_{u'v'} = 0.$$

The general solution is $\rho(u', v') = X(u') + Y(v')$ where $X, Y : \mathbb{D} \longrightarrow \mathbb{R}$.

Let $u := \int e^X du'$ and $v := \int e^Y dv'$. Define x and y by

$$x := \frac{u - v}{2},\ y := \frac{u + v}{2}.$$

Then

$$I = dudv = -dx^2 + dy^2.$$

□

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a simply connected flat Lorentz surface with globally defined isothermal coordinate system[3] (x, y) and the first fundamental form (17). Then

$$E := \langle \varphi_x, \varphi_x \rangle = -1,\ F := \langle \varphi_x, \varphi_y \rangle = 0,\ G := \langle \varphi_y, \varphi_y \rangle = 1.$$

Let N be a unit normal vector field on φ and let

$$\ell = \langle \varphi_{xx}, N \rangle,\ \mathfrak{m} = \langle \varphi_{xy}, N \rangle,\ \mathfrak{n} = \langle \varphi_{yy}, N \rangle.$$

Then the Gauß-Weingarten equations are given by

$$\varphi_{xx} = -\varphi + \ell N, \tag{19}$$

$$\varphi_{xy} = \mathfrak{m} N, \tag{20}$$

$$\varphi_{yy} = \varphi + \mathfrak{n} N, \tag{21}$$

$$N_x = \ell \varphi_x - \mathfrak{m} \varphi_y, \tag{22}$$

$$N_y = \mathfrak{m} \varphi_x - \mathfrak{n} \varphi_y. \tag{23}$$

[3] In Lorentzian case, the Riemann Mapping Theorem or Köbe Uniformization Theorem does not hold. So the global existence of isothermal coordinates is not guaranteed even in a simply connected Lorentzian 2-manifold. See, for example, [15] for details.

The Gauß-Mainardi-Codazzi equations, which are the integrability conditions for Gauß-Weingarten equations, are equivalent to

$$\mathfrak{m}^2 - \ell\mathfrak{n} = 1, \tag{24}$$

$$\mathfrak{m}_x = \ell_y,\ \mathfrak{n}_x = \mathfrak{m}_y. \tag{25}$$

The equations (25) guarantee the existence of potential functions ξ and η such that

$$\ell = \xi_x,\ \mathfrak{m} = \xi_y = \eta_x,\ \mathfrak{n} = \eta_y.$$

This also implies that there exists a potential ϕ such that $\xi = \phi_x$ and $\eta = \phi_y$. So $\ell, \mathfrak{m}, \mathfrak{n}$ can be written in terms of ϕ as

$$\ell = \phi_{xx},\ \mathfrak{m} = \phi_{xy} = \phi_{yx},\ \mathfrak{n} = \phi_{yy}.$$

The Gauß equation (15) then becomes the hyperbolic Monge-Ampère equation

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = -1. \tag{26}$$

The second fundamental form is given by

$$\begin{aligned} d\sigma^2 &= \ell dx^2 + 2\mathfrak{m}dxdy + \mathfrak{n}dy^2 \\ &= \phi_{xx}dx^2 + 2\phi_{xy}dxdy + \phi_{yy}dy^2. \end{aligned} \tag{27}$$

Note that the second fundamental form (27) determines a conformal structure, the so-called *second conformal structure*, in $\mathbb{D}$. To see this let

$$x' = x - \phi_x,\ y' = y + \phi_y.$$

Then by a straightforward calculation one obtains

$$\begin{aligned} dx &= \frac{1 + \phi_{yy}}{2 - \phi_{xx} + \phi_{yy}}dx' + \frac{\phi_{xy}}{2 - \phi_{xx} + \phi_{yy}}dy', \\ dy &= -\frac{\phi_{xy}}{2 - \phi_{xx} + \phi_{yy}}dx' + \frac{1 - \phi_{xx}}{2 - \phi_{xx} + \phi_{yy}}dy' \end{aligned} \tag{28}$$

and

$$\begin{aligned} d\sigma^2 &= \phi_{xx}dx^2 + 2\phi_{xy}dxdy + \phi_{yy}dy^2 \\ &= \frac{-dx'^2 + dy'^2}{2 - \phi_{xx} + \phi_{yy}} \\ &= \frac{dudv}{2 - \phi_{xx} + \phi_{yy}}, \end{aligned} \tag{29}$$

where $u := x' + y'$ and $v := -x' + y'$. Hence we see that (u, v) defines a null coordinate system with respect to the conformal structure in $\mathbb{D}$ determined by the second fundamental form. The differential operators $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial y'}$ are computed, in terms of the coordinates (x, y), to be:

$$\begin{aligned} \frac{\partial}{\partial x'} &= \frac{1+\phi_{yy}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial x} - \frac{\phi_{xy}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial y'} &= \frac{\phi_{xy}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial x} + \frac{1-\phi_{xx}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial y}. \end{aligned} \tag{30}$$

There are two things that need to be checked before we move on. One is whether the new coordinates (x', y') actually exist globally in $\mathbb{D}$ and the other is whether in (28) $2 - \phi_{xx} + \phi_{yy} \neq 0$ everywhere in $\mathbb{D}$. It turns out that:

Proposition 2. *The coordinates $x' = x - \phi_x,\ y' = y + \phi_y$ exist globally on $\mathbb{D}$ if and only if $2 - \phi_{xx} + \phi_{yy} \neq 0$.*

Proof. It follows from the Jacobian

$$\begin{aligned} \frac{\partial(x', y')}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} \\ &= 2 - \phi_{xx} + \phi_{yy}. \end{aligned}$$

□

In order to ensure the global existence of the coordinates (x', y') in $\mathbb{D}$, it is required that $2-\phi_{xx}+\phi_{yy} \neq 0$. Furthermore it is also required naturally that $2 - \phi_{xx} + \phi_{yy} > 0$ everywhere in order for the second fundamental form (29) to give rise to a conformal structure in $\mathbb{D}$. One may wonder if the quantity $2 - \phi_{xx} + \phi_{yy}$ has any geometric meaning. It does indeed:

$$2 - \phi_{xx} + \phi_{yy} = 2(H + 1) \tag{31}$$

where H is the mean curvature of φ. In other words, it is required for flat Lorentz surfaces under consideration to have $H > -1$. There is more to mention about this condition and it will be discussed in section 3.

3. A representation formula for flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$

In this section, it is shown that a flat Lorentz surface may be represented by a Lorentz holomorphic and a Lorentz anti-holomorphic data. We discuss this by means of the Lax system (12).

Suppose that $\mathbb{D}$ is a simply connected, oriented, 2-dimensional domain with globally defined null coordinate system (u', v'). Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a flat Lorentz surface with induced metric $ds^2_\varphi = e^\rho du'dv'$. Then by Proposition 1 we may assume that $\rho = 0$; hence the Gauß-Marnardi-Codazzi equations (15), (16) can be written as

$$H^2 - 1 = 4QR, \tag{32}$$

$$H_{u'} = 2Q_{v'},\ H_{v'} = 2R_{u'}. \tag{33}$$

From the Lax system (12), the 1-forms $F_1^{-1}dF_1$ and $F_2^{-1}dF_2$ are given by

$$\begin{aligned} F_1^{-1}dF_1 &= \begin{pmatrix} 0 & \frac{1}{2}(H+1)du' + Rdv' \\ -Qdu' - \frac{1}{2}(H-1)dv' & 0 \end{pmatrix}, \\ F_2^{-1}dF_2 &= \begin{pmatrix} 0 & Qdu' + \frac{1}{2}(H+1)dv' \\ -\frac{1}{2}(H-1)du' - Rdv' & 0 \end{pmatrix}. \end{aligned} \tag{34}$$

The Mainardi-Codazzi equations (33) imply that the nonzero entries of $F_1^{-1}dF_1$ and $F_2^{-1}dF_2$ are exact, i.e. there exist functions $u, \xi, v, \zeta : \mathbb{D}(u', v') \longrightarrow \mathbb{E}^2_1$ such that

$$F_1^{-1}dF_1 = \begin{pmatrix} 0 & du \\ d\xi & 0 \end{pmatrix} \text{ and } F_2^{-1}dF_2 = \begin{pmatrix} 0 & dv \\ d\zeta & 0 \end{pmatrix}. \tag{35}$$

Proposition 3. *The functions (u, v) constitute globally defined null coordinates of $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ if and only if the mean curvature $H \neq -1$.*

Proof. It follows from the Jacobian

$$\begin{aligned} \frac{\partial(u,v)}{\partial(u',v')} &= \begin{vmatrix} \frac{\partial u}{\partial u'} & \frac{\partial u}{\partial v'} \\ \frac{\partial v}{\partial u'} & \frac{\partial v}{\partial v'} \end{vmatrix} \\ &= \begin{vmatrix} \frac{1}{2}(H+1) & R \\ Q & \frac{1}{2}(H+1) \end{vmatrix} \\ &= \frac{1}{2}(H+1) \end{aligned}$$

by the Gauß equation (5). ☐

Remark 1. In order to consider flat Lorentz surfaces with respect to the new null coordinate system (u, v), it is required that $H > -1$.

Remark 2 (The Lawson correspondence). The *Lawson correspondence* is a correspondece between Lorentz surfaces in three different Lorentzian space-forms $\mathbb{E}^3_1$, $\mathbb{S}^3_1$, and $\mathbb{H}^3_1(-1)$, that satisfy the same Gauß-Mainardi-Codazzi equations. As a result they share a number of properties in common even though they are residing in different space-forms. For that reason they are called *cousins* (in the sense of the Lawson correspondence). For details see appendix A of [9]. The Gauß equations in the three Lorentzian space-forms are given by:

$$\begin{aligned} &\rho_{u'v'} + \frac{1}{2}H_e^2 e^{\rho} - 2QRe^{-\rho} = 0 \text{ (Minkowski 3-space } \mathbb{E}^3_1) \\ &\rho_{u'v'} + \frac{1}{2}(H_s^2+1) e^{\rho} - 2QRe^{-\rho} = 0 \text{ (de Sitter 3-space } \mathbb{S}^3_1) \\ &\rho_{u'v'} + \frac{1}{2}(H_h^2-1) e^{\rho} - 2QRe^{-\rho} = 0 \text{ (anti-de Sitter 3-space } \mathbb{H}^3_1(-1)) \end{aligned}$$

where H_e, H_s, and H_h denote respectively the mean curvature in each space-form. Clearly a Lorentz surface in $\mathbb{H}^3_1(-1)$ has cousins in other Lorentzian space-forms if and only if $H_h^2 - 1 \geq 0$, or equivalently, $H_h \leq -1$ or $H_h \geq 1$. That is flat Lorentz surfaces in $\mathbb{H}_1^3(-1)$ with $-1 < H_h < 1$ do not have cousins in any Lorentzian space-form, whilst those with $H_h \geq 1$ do. Flat Lorentz surfaces in $\mathbb{H}_1^3(-1)$ with $H_h = 1$ are particularly interesting because they resemble horospheres in hyperbolic 3-space: they are flat and totally umbilic, or equivalently they have constant (hyperbolic) Gauß map[4]. Their cousins in $\mathbb{E}_1^3$ are of course timelike planes and there are no cousins[5] in $\mathbb{S}_1^3$.

The following proposition tells that $F_1^{-1}dF_1$ is a Lorentz holomorphic 1-form and $F_2^{-1}dF_2$ is a Lorentz anti-holomorphic 1-form.

Proposition 4. *$\frac{\partial\xi}{\partial v} = 0$ and $\frac{\partial\zeta}{\partial u} = 0$ i.e. ξ is Lorentz holomorphic and ζ is Lorentz anti-holomorphic.*

Proof.

$$\begin{aligned}
\frac{\partial\xi}{\partial v} &= \frac{\partial\xi}{\partial u'}\frac{\partial u'}{\partial v} + \frac{\partial\xi}{\partial v'}\frac{\partial v'}{\partial v} \\
&= -Q\frac{\partial u'}{\partial v} - \frac{1}{2}(H-1)\frac{\partial v'}{\partial v} \\
&= -\frac{\partial v}{\partial u'}\frac{\partial u'}{\partial v} + \left(1 - \frac{\partial v}{\partial v'}\right)\frac{\partial v'}{\partial v} \\
&= \frac{\partial v'}{\partial v} - 1.
\end{aligned}$$

Thus $\frac{\partial\xi}{\partial v} = 0$ if and only if $\frac{\partial v'}{\partial v} = 1$.

Now we have the system of linear equations:

$$\begin{cases}
1 = \dfrac{\partial v'}{\partial u}\dfrac{\partial u}{\partial v'} + \dfrac{\partial v'}{\partial v}\dfrac{\partial v}{\partial v'} \\
\quad = R\dfrac{\partial v'}{\partial u} + \dfrac{1}{2}(H+1)\dfrac{\partial v'}{\partial v}, \\
0 = \dfrac{\partial v'}{\partial u}\dfrac{\partial u}{\partial u'} + \dfrac{\partial v'}{\partial v}\dfrac{\partial v}{\partial u'} \\
\quad = \dfrac{1}{2}(H+1)\dfrac{\partial v'}{\partial u} + Q\dfrac{\partial v'}{\partial v}
\end{cases}.$$

Solving this system we obtain $\frac{\partial v'}{\partial v} = 1$; hence $\frac{\partial\xi}{\partial v} = 0$.

It can be similarly shown that $\frac{\partial\zeta}{\partial u} = 0$. □

Let $f = \frac{\partial\xi}{\partial u}$ and $g = \frac{\partial\zeta}{\partial v}$. Then $d\xi = f du$ and $d\zeta = g dv$ since ξ and ζ are, respectively, Lorentz holomorphic and Lorentz anti-holomorphic. The 1-forms

---

[4]The notion of hyperbolic Gauß map will be introduced in section 4.

[5]$\mathbb{S}_1^3$ does not admit timelike surfaces that are analogues of horspheres but it does admit spacelike surfaces that are analogues of horospheres.

$F_1^{-1}dF_1$ and $F_2^{-1}dF_2$ are then written as

$$F_1^{-1}dF_1 = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} du \text{ and } F_2^{-1}dF_2 = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} dv.$$

The induced metric ds^2_φ is computed[6] to be

$$\begin{aligned} ds^2_\varphi &= \varphi^*(ds^2) \\ &= \langle d\varphi, d\varphi \rangle \\ &= \langle d(F_1 F_2^t), d(F_1 F_2^t) \rangle \\ &= -\det\{F_1^{-1}dF_1 + (F_2^{-1}dF_2)^t\} \\ &= f du^2 + g dv^2 + (1+fg) du dv. \end{aligned}$$

The functions f and g are given by

$$\begin{aligned} f &= \frac{\partial \xi}{\partial u'}\frac{\partial u'}{\partial u} + \frac{\partial \xi}{\partial v'}\frac{\partial v'}{\partial u} \\ &= -\frac{2Q}{H+1}, \\ g &= \frac{\partial \zeta}{\partial u'}\frac{\partial u'}{\partial u} + \frac{\partial \zeta}{\partial v'}\frac{\partial v'}{\partial u} \\ &= -\frac{2R}{H+1}. \end{aligned}$$

The ∂' and ∂'' forms[7] du' and dv' are given by

$$\begin{aligned} du' &= \frac{\partial u'}{\partial u}du + \frac{\partial u'}{\partial v}dv \\ &= du + g dv, \\ dv' &= \frac{\partial v'}{\partial u}du + \frac{\partial v'}{\partial v}dv \\ &= f du + dv. \end{aligned}$$

Finally the second fundamental form $d\sigma^2$ is computed to be

$$\begin{aligned} d\sigma^2 &= Q du'^2 + R dv'^2 + H ds^2_\varphi \\ &= (1 - fg) du dv. \end{aligned}$$

Therefore we have the following theorem holds:

Theorem 5. *Let $\mathbb{D}$ be a simply connected, oriented, 2-dimensional domain with globally defined null coordinate system (u', v'). Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a flat Lorentz surface with induced metric $ds^2_\varphi = du'dv'$, mean curvature*

[6] It can be also obtained by calculating $du'dv'$.

[7] They are analogues of $(1,0)$ and $(0,1)$ forms in complex analysis.

$H > -1$, and Hopf differential $Qdu'^2 + Rdv'^2$. Then there exist globally defined null coordinate system (u, v) in $\mathbb{D}$ such that

$$du = \frac{1}{2}(H+1)du' + Rdv' \text{ and } dv = Qdu' + \frac{1}{2}(H+1)dv'.$$

Define $f, g : \mathbb{D} \longrightarrow \mathbb{E}^2_1$ by

$$f = -\frac{2Q}{H+1} \text{ and } g = -\frac{2R}{H+1}, \text{ respectively.}$$

Then f is a Lorentz holomorphic function and g is a Lorentz anti-holomorphic function with respect to the null coordinate system (u, v), and that $fg < 1$. The flat Lorentz surface $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ may be descibed by

$$\varphi = F_1 F_2^t$$

where $F_1, F_2 : \mathbb{D} \longrightarrow \mathrm{SL}(2, \mathbb{R})$ are immersions that satisfy the system of decoupled ODEs:

$$F_1^{-1}dF_1 = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} du \text{ and } F_2^{-1}dF_2 = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} dv.$$

The first and the second fundamental forms are, respectively, given in terms of f and g by

$$ds^2_\varphi = fdu^2 + (1 + fg)dudv + gdv^2, \tag{36}$$

$$d\sigma^2 = (1 - fg)dudv. \tag{37}$$

Remark 3. As seen in (37) the second fundamental form $d\sigma^2$ determines a conformal structure with respect to the null coordinate system (u, v).

The converse of Theorem 5 also holds, namely

Theorem 6. *Let $\mathbb{D}$ be an oriented 2-dimensional domain with globally defined null coordinate system (u', v'). Suppose that $F_1, F_2 : \mathbb{D} \longrightarrow \mathrm{SL}(2, \mathbb{R})$ are immersions that satisfy the Lax system (12). If there exist a null coordinate system (u, v) globally defined in $\mathbb{D}$ and functions $f, g : \mathbb{D} \longrightarrow \mathbb{E}^2_1$ such that*

$$F_1^{-1}dF_1 = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} du \text{ and } F_2^{-1}dF_2 = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} dv$$

where f is Lorentz holomorphic, g is Lorentz anti-holomorphic with respect to (u, v), and that $fg < 1$, then

$$\varphi := F_1 F_2^t : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$$

is a flat Lorentz surface whose first and second fundamental forms are given by (36) and (37) respectively. Furthermore $H > -1$ and

$$f = -\frac{2Q}{e^\rho(H+1)}, \quad g = -\frac{2R}{e^\rho(H+1)}$$

where ρ is a constant and $Qdu'^2 + Rdv'^2$ is the Hopf differential of φ.

Proof. Suppose that $F_1, F_2 : \mathbb{D} \longrightarrow \mathrm{SL}(2,\mathbb{R})$ satisfy the Lax system (12). Then

(38)
$$
\begin{aligned}
F_1^{-1}dF_1 &= F_1^{-1}\{(F_1)_{u'}du' + (F_1)_{\mathbf{v}'}dv'\}\\
&= \begin{pmatrix} \frac{\rho_{u'}}{4}du' - \frac{\rho_{v'}}{4}dv' & \frac{1}{2}e^{\rho/2}(H+1)du' + e^{-\rho/2}Rdv' \\ -e^{-\rho/2}Qdu' - \frac{1}{2}e^{\rho/2}(H-1)dv' & -\frac{\rho_{u'}}{4}du' + \frac{\rho_{v'}}{4}dv' \end{pmatrix}\\
&= \begin{pmatrix} 0 & du \\ d\xi & 0 \end{pmatrix}
\end{aligned}
$$

and

(39)
$$
\begin{aligned}
F_2^{-1}dF_2 &= F_2^{-1}\{(F_2)_{u'}du' + (F_2)_{v'}dv'\}\\
&= \begin{pmatrix} -\frac{\rho_{u'}}{4}du' + \frac{\rho_{v'}}{4}dv' & e^{-\rho/2}Qdu' + \frac{1}{2}e^{\rho/2}(H+1)dv' \\ -\frac{1}{2}e^{\rho/2}(H-1)du' - e^{-\rho/2}Rdv' & \frac{\rho_{u'}}{4}du' - \frac{\rho_{v'}}{4}dv' \end{pmatrix}\\
&= \begin{pmatrix} 0 & dv \\ d\zeta & 0 \end{pmatrix}
\end{aligned}
$$

where $d\xi = fdu$ and $d\zeta = gdv$. Clearly $\rho = c$, a constant. Since ξ is Lorentz holomorphic with respect to (u, v),

$$
\begin{aligned}
\frac{\partial \xi}{\partial u'} &= \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial u'} = \frac{1}{2}e^{\frac{c}{2}}(H+1)\frac{\partial \xi}{\partial u},\\
\frac{\partial \xi}{\partial v'} &= \frac{\partial \xi}{\partial u}\frac{\partial u}{\partial v'} = e^{-\frac{c}{2}}R\frac{\partial \xi}{\partial u}.
\end{aligned}
\tag{40}
$$

From (38) we also find

$$
\frac{\partial \xi}{\partial u'} = -e^{-\frac{c}{2}}Q,\ \frac{\partial \xi}{\partial v'} = -\frac{1}{2}e^{\frac{c}{2}}(H-1). \tag{41}
$$

Combining (40) and (41) we obtain the equation

$$
(H^2 - 1 - 4e^{-2c}QR)\frac{\partial \xi}{\partial u} = K\frac{\partial \xi}{\partial u} = 0. \tag{42}
$$

If $\frac{\partial \xi}{\partial u} = 0$ then $d\xi = 0$ so $Q = 0$ and $H = 1$. This implies that $K = 0$ by the Gauß equation (5). Moreover this is the case that $\varphi = F_1F_2^t$ is an analogue of horospheres. If $\frac{\partial \xi}{\partial u} \neq 0$ then again $K = 0$ by (42). Hence in either case $\varphi = F_1F_2^t$ is a flat Lorentz surface in $\mathbb{H}^3_1(-1)$.

The following identities can be calculated from (38) and (39):

$$
\begin{aligned}
f &= -\frac{2Q}{e^c(H+1)},\ g = -\frac{2R}{e^c(H+1)},\\
du' &= e^{-2/c}du + e^{-c/2}gdv,\ dv' = e^{-c/2}fdu + e^{-c/2}dv.
\end{aligned}
$$

Using these identities we can write the first and the second fundamental forms in terms of f and g as:

$$\begin{aligned} ds^2_\varphi &= e^c du' dv' \\ &= f du^2 + (1+fg) du dv + g dv^2, \\ d\sigma^2 &= Q du'^2 + H ds^2_\varphi + R dv'^2 \\ &= (1-fg) du dv. \end{aligned}$$

Since $1 - fg = \frac{2}{H+1} > 0$, $H > -1$. □

4. Flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$ and the hyperbolic Gauss map

In the study of flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$, the hyperbolic Gauß map[8] plays an important role. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a Lorentz surface. At each point $p \in \mathbb{D}$, the oriented normal geodesic in $\mathbb{H}^3_1(-1)$ emanating from $\varphi(p)$, which is tangent to the normal vector $N(p)$ meets the null cone $\mathbb{N}^3 = \{\mathbf{u} \in \mathbb{E}^4_2 : \langle \mathbf{u}, \mathbf{u} \rangle = 0\}$ at exactly two points $[\varphi + N](p)$ and $[\varphi - N](p)$ in $\mathbb{N}^3$. Here, $[\varphi \pm N]$ denotes the null lines spanned by the null vectors $\varphi \pm N$. The orientation of φ allows us to name $[\varphi+N](p)$ the initial point and $[\varphi-N](p)$ the terminal point. The maps $G_+, G_- : \mathbb{D} \longrightarrow \mathbb{N}^3$ defined by $G_+(p) = [\varphi + N](p)$ and $G_-(p) = [\varphi - N](p)$, resp. are called the hyperbolic Gauss map of φ. In this paper, we are particularly interested in the hyperbolic Gauß map $G_- = [\varphi - N]$.

The equation of null cone $\mathbb{N}^3$ in $\mathbb{E}^4_2$ is given by

$$-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = 0. \tag{43}$$

Assume[9] that $x_0 \neq 0$. By means of nonhomogeneous coordinates

$$\begin{aligned} [\varphi \pm N] &= [x^0, x^1, x^2, x^3] \\ &= \left[1, \frac{x^1}{x^0}, \frac{x^2}{x^0}, \frac{x^3}{x^0}\right]. \end{aligned}$$

Let $\mathbb{N}^3_+ := \{(x^0, x^1, x^2, x^3) \in \mathbb{N}^3 : x^0 > 0\}$, the future[10] null cone and $\mathbb{N}^3_- := \{(x^0, x^1, x^2, x^3) \in \mathbb{N}^3 : x^0 < 0\}$, the past null cone. The multiplicative group $\mathbb{R}^+$ acts on $\mathbb{N}^3_+$ and $\mathbb{N}^3_-$, resp. by scalar multiplication. Let us denote by $\mathbb{N}^3_+/\mathbb{R}^+$ and $\mathbb{N}^3_-/\mathbb{R}^+$ the orbit spaces of $\mathbb{N}^3_+$ and $\mathbb{N}^3_-$, resp.

---

[8]The hyperbolic Gauß map was first introduced by C. Epstein [5] and was also used by R. L. Bryant in the study of constant mean curvature one surfaces in hyperbolic 3-space [1]. We will keep the name in our paper to avoid verbosity such as saying that an analogue of hyperbolic Gauß map.

[9]If $x^0 = 0$ then use x^1 instead. If both $x^0 = x^1 = 0$ then clearly $(x^0, x^1, x^2, x^3) = (0, 0, 0, 0)$.

[10]The notion of future and past may be ambiguous in $\mathbb{E}^4_2$. Recall that there are two coordinate times in $\mathbb{E}^4_2$.

Let $(x^0, x^1, x^2, x^3) \in \mathbb{N}^3_+$ $((x^0, x^1, x^2, x^3) \in \mathbb{N}^3_-)$. Then the equation of null cone (43) is equivalent to

$$-\left(\frac{x^1}{x^0}\right)^2 + \left(\frac{x^2}{x^0}\right)^2 + \left(\frac{x^3}{x^0}\right)^2 = 1,$$

i.e. $\left(\frac{x^1}{x^0}, \frac{x^2}{x^0}, \frac{x^2}{x^0}\right) \in \mathbb{S}^2_1$ where $\mathbb{S}^2_1$ denotes the *pseudosphere* in Minkwoski 3-space $\mathbb{E}^3_1$:

$$\mathbb{S}^2_1 = \{(\xi^0, \xi^1, \xi^2) \in \mathbb{E}^3_1 : -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 = 1\}.$$

If there is an observer[11] at the origin (the event, physically speaking), light rays through his eye correspond to null lines through the origin. The past null directions then constitute the field of vision of the observer which is the pseudosphere $\mathbb{S}^2_1$. In other words, $\mathbb{S}^2_1$ is what the observer sees provided he is stationary relative to the frame (x^0, x^1, x^2, x^3). This is an analogue of what is called the *celestial sphere*[12] in Minkowski space-time. Interesting readers may consult, for instance, the book [13] by Roger Penrose and Wolfgang Rindler for details.

The map[13] $f : \mathbb{N}^3_+$ $(\mathbb{N}^3_-) \longrightarrow \mathbb{S}^2_1$ defined by

$$f(x^0, x^1, x^2, x^3) = \left(\frac{x^1}{x^0}, \frac{x^2}{x^0}, \frac{x^3}{x^0}\right) \in \mathbb{S}^2_1$$

is a surjective identification map; hence $\mathbb{N}^3_+/\mathbb{R}^+$ $(\mathbb{N}^3_-/\mathbb{R}^+)$ is homeomorphic to $\mathbb{S}^2_1$. Furthermore, $\mathbb{N}^3_+/\mathbb{R}^+$ $(\mathbb{N}^3_-/\mathbb{R}^+)$ is diffeomorphic to $\mathbb{S}^2_1$.

Let $\mathcal{N} = (0, 0, 1)$ and $\mathcal{S} = (0, 0, -1) \in \mathbb{S}^2_1$ be the *north pole* and the *south pole* of $\mathbb{S}^2_1$. Let $\wp_+ : \mathbb{S}^2_1 \backslash \{\xi^3 = -1\} \longrightarrow \mathbb{E}^2_1 \backslash \mathbb{H}^1_0$ be the stereographic projection from the south pole $\mathcal{S}$, where $\mathbb{H}^1_0 = \{(\xi^1, \xi^2) \in \mathbb{E}^2_1 : -(\xi^1)^2 + (\xi^2)^2 = -1\}$ is the hyperbola in $\mathbb{E}^2_1$. Then

$$\begin{aligned} \wp_+(\xi^1, \xi^2, \xi^3) &= \left(\frac{\xi^1}{1+\xi^3}, \frac{\xi^2}{1+\xi^3}\right) \\ &= \left(\frac{\xi^1+\xi^2}{1+\xi^3}, \frac{-\xi^1+\xi^2}{1+\xi^3}\right) \in \mathbb{E}^2_1(u, v), \end{aligned} \tag{44}$$

where (u, v) is a null coordinate system in $\mathbb{E}^2_1$. The second equality is an abuse of notation. It is actually an identification due to a diffeomorphism.

[11] The semi-Euclidean space $\mathbb{E}^4_2$ is not actually a physical space-time, though it provides an interesting physical model, anti-de Sitter 3-space, in string theory with regard to the famous AdS/CFT conjecture by J. Maldacena [11].

[12] The celestial sphere can be identified with the Riemann sphere $\mathbb{S}^2$.

[13] The map $f : \mathbb{N}^3_- \longrightarrow \mathbb{S}^2_1$ is an analogue of what is called the *sky mapping* in Minkowski space-time.

Let $\wp_- : \mathbb{S}^2_1 \setminus \{\xi^3 = 1\} \longrightarrow \mathbb{E}^2_1 \setminus \mathbb{H}^1_0$ be the stereographic projection from the north pole $\mathcal{N}$. Then

$$\begin{aligned}\wp_-(\xi^1, \xi^2, \xi^3) &= \left(\frac{\xi^1}{1-\xi^3}, \frac{\xi^2}{1-\xi^3}\right) \\ &= \left(\frac{\xi^1+\xi^2}{1-\xi^3}, \frac{-\xi^1+\xi^2}{1-\xi^3}\right) \in \mathbb{E}^2_1(u,v).\end{aligned} \tag{45}$$

Hence the projected hyperbolic Gauß map is mapped into $\mathbb{E}^2_1(u,v)$.

Let $F_1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{13} & F_{14} \end{pmatrix}$ and $F_2 = \begin{pmatrix} F_{21} & F_{22} \\ F_{23} & F_{24} \end{pmatrix}$ where $F = (F_1, F_2)$ is a coordinate frame. Then by (11)

$$\begin{aligned}\varphi - N &= F_1(\mathbf{1} - \mathbf{k}')F_2^t \\ &= 2F_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} F_2^t \\ &= 2\begin{pmatrix} F_{12}F_{22} & F_{12}F_{24} \\ F_{14}F_{22} & F_{14}F_{24} \end{pmatrix} \\ &= 2\begin{pmatrix} F_{12} \\ F_{14} \end{pmatrix}\begin{pmatrix} F_{22} \\ F_{24} \end{pmatrix}^t.\end{aligned}$$

Thus the hyperbolic Gauß map $[\varphi - N]$ is written as

$$[\varphi - N] = \left[\begin{pmatrix} F_{12} \\ F_{14} \end{pmatrix}\begin{pmatrix} F_{22} \\ F_{24} \end{pmatrix}^t\right].$$

By the identification (6), the projected hyperbolic Gauß map is given by

$$[\varphi - N] \overset{\wp_-}{=} \left(\frac{F_{12}}{F_{14}}, \frac{F_{22}}{F_{24}}\right) \in \mathbb{E}^2_1(u,v) \tag{46}$$

and

$$[\varphi - N] \overset{\wp_+}{=} \left(\frac{F_{24}}{F_{22}}, \frac{F_{14}}{F_{12}}\right) \in \mathbb{E}^2_1(u,v). \tag{47}$$

By (22), (23), and (30) one obtains

$$(\varphi - N)_{x'} = \varphi_x,\ (\varphi - N)_{y'} = \varphi_y \tag{48}$$

and thereby

$$(\varphi - N)_u = \frac{1}{2}(\varphi_x + \varphi_y),\ (\varphi - N)_v = \frac{1}{2}(-\varphi_x + \varphi_y).$$

It then follows that

$$\begin{aligned}&\langle(\varphi - N)_u, (\varphi - N)_u\rangle = \langle(\varphi - N)_v, (\varphi - N)_v\rangle = 0, \\ &\langle(\varphi - N)_u, (\varphi - N)_v\rangle = \frac{1}{2}.\end{aligned} \tag{49}$$

Let $d\rho^2$ denote the induced metric on $\mathbb{N}^3$. Then the pullback of $d\rho^2$ by $\varphi - N$ is

$$\begin{aligned} d\rho^2_{\varphi-N} &:= (\varphi - N)^*(d\rho^2) \\ &= \langle d(\varphi - N), d(\varphi - N)\rangle \\ &= dudv \end{aligned}$$

by (49). This means that the hyperbolic Gauß map $[\varphi - N]$ is conformal with respect to the second conformal structure in $\mathbb{D}$. In fact, more can be said about conformal hyperbolic Gauß maps.

Theorem 7. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a Lorentz surface with unit normal vector field N and mean curvature $H \geq 1$. The hyperbolic Gauß map $[\varphi - N]$: $\mathbb{D} \longrightarrow \mathbb{S}^2_1$ is conformal with respect to the second fundamental form if and only if φ is flat or totally umbilic. Here the pseudosphere $\mathbb{S}^2_1$ is viewed as the orbit space $\mathbb{N}^3_+/\mathbb{R}^+$ or $\mathbb{N}^3_-/\mathbb{R}^+$.*

Proof. Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a Lorentz surface with induced metric $ds^2_\varphi = e^\rho du'dv'$ where (u', v') is a globally defined null coordinate system in $\mathbb{D}$. Let $\mathfrak{s} := (\varphi, \varphi_{u'}, \varphi_{v'}, N)$. The $\mathfrak{s}$ defines a moving frame on φ and satisfy the Gauß-Weingarten equations:

$$\mathfrak{s}_{u'} = \mathfrak{s}\mathcal{U},\ \mathfrak{s}_{v'} = \mathfrak{s}\mathcal{V}$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\rho & 0 \\ 1 & \rho_{u'} & 0 & -H \\ 0 & 0 & 0 & -2Qe^{-\rho} \\ 0 & Q & \frac{1}{2}e^\rho H & 0 \end{pmatrix},\ \mathcal{V} = \begin{pmatrix} 0 & \frac{1}{2}e^\rho & 0 & 0 \\ 0 & 0 & 0 & -2Re^{-\rho} \\ 1 & 0 & \rho_{v'} & -H \\ 0 & \frac{1}{2}e^\rho H & R & 0 \end{pmatrix}.$$

Using the Gauß-Weingarten equations, one can calculate

$$\begin{aligned} d\rho^2 &= \langle d(\varphi - N), d(\varphi - N)\rangle \\ &= -Kds^2_\varphi + 2(H+1)d\sigma^2 \\ &= \left[2(H+1) - \frac{K}{H}\right] d\sigma^2 + \frac{K\mathcal{Q}}{H}. \end{aligned}$$

Therefore $[\varphi - N]$ is conformal with respect to the second fundamental form $d\sigma^2$ if and only if φ is flat or totally umbilic. □

Remark 4. If $H = -1$ then $d\rho^2 = \langle d(\varphi - N), d(\varphi - N)\rangle = Kd\sigma^2 - K\mathcal{Q}$. If $K = 0$ in addition then $d\rho^2$ is degenerate and $d(\varphi - N) = 0$, i.e. the hyperbolic Gauß map $[\varphi - N]$ is constant. This is the case when φ is an analogue of horosphere.

In light of theorem 7, one may wonder if there is any connection between the flatness of a Lorentz surfaces and the holomorphicity of the hyperbolic Gauß map.

Theorem 8. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a Lorentz surface with mean curvature $H \geq 1$. φ is flat or totally umbilic if and only if the first and the second coordinates of the projected hyperbolic Gauß map* (46) *are Lorentz anti-holomorphic and Lorentz holomorphic, respectively, with respect to null coordinates (u, v) determined by the second fundamental form.*

Proof. If φ is totally umbilic, then the first and the second coordinates of the projected Gauß map are Lorentz anti-holomorphic and Lorentz-holomorphic with respect to null coordinates determined by Lorentz isothermal coordinates as shown in section 13 of [9]. Since $II = HI$, the same is true for null coordinates (u, v) determined by the second fundamental form.

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{H}^3_1(-1)$ be a Lorentz surface with the first fundamental form $ds_\varphi = e^\rho du'dv'$. Suppose that (u, v) is another null coordinate system globally defined in $\mathbb{D}$. From the Lax system (12) we obtain

$$(F_{12})_v F_{14} - F_{12}(F_{14})_v = \frac{1}{2}e^{\rho/2}(H+1)\frac{\partial u'}{\partial v} + e^{-\rho/2}R\frac{\partial v'}{\partial v}, \tag{50}$$

$$(F_{22})_u F_{24} - F_{22}(F_{24})_u = e^{-\rho/2}Q\frac{\partial u'}{\partial u} + \frac{1}{2}e^{\rho/2}(H+1)\frac{\partial v'}{\partial u}. \tag{51}$$

If φ is flat and (u, v) is a null coordinate system determined by the second fundamental form, then without loss of generality we may take

$$\begin{aligned}
\frac{\partial u'}{\partial v} &= -\frac{2Re^{-\rho/2}}{e^\rho(H+1)},\ \frac{\partial v'}{\partial v} = e^{-\rho/2},\\
\frac{\partial u'}{\partial u} &= e^{-\rho/2},\ \frac{\partial v'}{\partial u} = -\frac{2Qe^{-\rho/2}}{e^\rho(H+1)}.
\end{aligned}$$

It then follows that $\left(\frac{F_{12}}{F_{14}}\right)_v = 0$ and $\left(\frac{F_{22}}{F_{24}}\right)_u = 0$.

In order to show the "if" part of the statement, suppose that the first fundamental form is given by $I = e^\rho du'dv'$ while the second fundamental form is given by $II = e^\lambda dudv$. If the first and the second coordinates of the projected hyperbolic Gauß map are, respectively, Lorentz anti-holomorphic and Lorentz holomorphic, then by (50) and (51)

$$\begin{aligned}
\frac{1}{2}e^{-\rho/2}(H+1)\frac{\partial u'}{\partial u} + e^{-\rho/2}R\frac{\partial v'}{\partial u} &= 0,\\
e^{-\rho/2}Q\frac{\partial u'}{\partial v} + \frac{1}{2}e^{-\rho/2}(H+1)\frac{\partial v'}{\partial v} &= 0.
\end{aligned}$$

Since $\frac{\partial(u',v')}{\partial(u,v)} \neq 0$,

$$\begin{aligned}
\frac{\partial v}{\partial u'} &= \frac{1}{2}e^{-\rho/2}(H+1),\ \frac{\partial v}{\partial v'} = e^{-\rho/2}R,\\
\frac{\partial u}{\partial u'} &= e^{-\rho/2}Q,\ \frac{\partial u}{\partial v'} = \frac{1}{2}e^{-\rho/2}(H+1).
\end{aligned}$$

Hence

$$\begin{aligned} II =& e^\lambda du dv\\ =& \frac{1}{2}e^\lambda Q(H+1)du'^2+\frac{1}{4}e^\lambda e^\rho[2H(H+1)-K]du'dv'+\frac{1}{2}e^\lambda R(H+1)dv'^2. \end{aligned}$$

Comparing this with $II = Qdu'^2 + He^\rho du'dv' + Rdv'^2$, we obatin

$$\begin{aligned} [e^\lambda(H+1)-2]Q &= 0,\\ [e^\lambda(H+1)-2]R &= 0,\\ 2[H+1-2e^{-\lambda}]H &= K. \end{aligned}$$

If $H = 2e^{-\lambda} - 1$ then $K = 0$. Otherwise $Q = R = 0$. This completes the proof. □

Remark 5. Since we require that $H \geq 1$, λ must satisfy $\lambda \leq 0$.

Remark 6. If φ is both flat and totally umbilic then by the Gauß equation (5) $H = 1$, i.e. $\lambda = 0$.

5. Flat Lorentz Surfaces in $\mathbb{H}^3_1(-1)$ and Gravitational Instantons

Any self-dual or anti-self-dual curvature 2-form gives a vanishing Ricci tensor, so any metric yielding a self-dual or anti-self-dual connection satisfies the Euclidean Einstein's field equations. There are also self-dual or anti-self-dual solutions of the Einstein's equations that have additional property that the metric approaches a flat metric at infinity. Solutions satisfying such property are called *asymptotically locally Euclidean* (abbreviated ALE) metrics. This hints that a certain compactness of the base manifold is expected to insure the existece of ALE metrics. Since Yang-Mills gauge potential approaches a pure gauge at infinity, ALE metrics also closely resemble the Yang-Mills instantons. For this reason, ALE metrics are also called *gravitational instantons.* Interesting readers may consult [2], [3], and [4] for details and examples of gravitational instantons.

The Euclidean Einstein's field equations for anti-self-dual gravitational fields reduce to a complex elliptic Monge-Ampére equation ([16], [17]). In [12], Y. Nutku considered the following equation

$$(\partial\bar{\partial}u)^2 = C * \mathbf{1} \tag{52}$$

on a complex 2-manifold M, which is rather a simpler form of the complex Monge-Ampére equation given in [16] and [17]. Here u is the Kähler potential, $\partial, \bar{\partial}$ are the Dolbeault operators, $*\mathbf{1}$ is the normalized volume element of M, and C is a constant[14].

[14] The constant C is different from the constant given in [12]. The way the equation is written in [12] results an extra negative sign.

In [12], the author studied a naive 2-dimensional reduction of (52) that the Kähler potential u depends only on two real coordinates t and x, where $\zeta_1 = t + iz$ and $\zeta_2 = x + iy$ are two complex local coordinates of M. So the resulting Kähler potential is translation-invariant in the z, y-directions when M is viewed as a 4-dimensional differentiable manifold.

Since any complex manifold admits a Hermitian metric, let us denote by g a Hermitian metric of M. As is well-known a Hermitian metric on a complex manifold locally takes the form

$$g = g_{\mu\bar{\nu}} d\zeta^\mu \otimes d\bar{\zeta}^\nu + g_{\bar{\mu}\nu} d\bar{\zeta}^\mu \otimes d\zeta^\nu, \tag{53}$$

where $\mu, \nu = 1, 2$. Suppose that g is a Kähler metric, then locally $g_{\mu\bar{\nu}}$ is given by

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} u, \tag{54}$$

where u is a function called the *Kähler potential* of the Kähler metric g.

Now the 2-dimensional reduction $u = u(t, x)$ yields the Kähler metric[15]

$$g = u_{tt}(dt^2 + dz^2) + 2u_{tx}(dtdx + dydz) + u_{xx}(dx^2 + dy^2) \tag{55}$$

and the Kähler 2-form $\Omega = i\partial\bar{\partial}u$ of the metric g is given by

$$\Omega = u_{tt} dt \wedge dz + u_{tx}(dt \wedge dy + dx \wedge dz) + u_{xx} dx \wedge dy. \tag{56}$$

From this reduction the equation (52) yields the 2-dimensional Monge-Ampére equation

$$u_{tt}u_{xx} - u_{tx}^2 = \kappa, \tag{57}$$

where κ is a constant. The constant κ is determined by choosing an appropriate value of C, so that the Monge-Ampére equation (57) is elliptic if $\kappa > 0$ and hyperbolic if $\kappa < 0$. The Monge-Ampére equation (57) may be regarded as a 2-dimensional reduction of the Euclidean Einstein's field equation that governs anti-self-dual solutions.

Suppose that $\phi(t, x)$ is a solution to the equation

$$\phi_{tt}(1 + (\phi_x)^2) - 2\phi_t\phi_x\phi_{tx} + \phi_{xx}(\kappa + (\phi_t)^2) = 0. \tag{58}$$

Then the following equations hold:

$$\begin{aligned} -\frac{\partial}{\partial t}\left(\frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}\right) + \frac{\partial}{\partial x}\left(\frac{\kappa + \phi_t^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}\right) &= 0, \\ \frac{\partial}{\partial t}\left(\frac{1 + \phi_x^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}\right) - \frac{\partial}{\partial x}\left(\frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}\right) &= 0. \end{aligned} \tag{59}$$

[15] with a scaling

This implies that there exists a solution $u(t,x)$ to the Monge-Ampére equation (57) such that

$$u_{tt} = \frac{\kappa + (\phi_t)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}},\ u_{tx} = \frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}},$$
(60)
$$u_{xx} = \frac{1 + (\phi_x)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}.$$

The transformation in (60) is a slight variation of what is discussed in K. Jörgens's 1954 paper [8]. Using this transformation we can write a metric for a class of anti-self-dual gravitational instantons:

(61)
$$ds^2 = \frac{\kappa + (\phi_t)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}(dt^2 + dz^2) + 2\frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}(dtdx + dydz)$$
$$+ \frac{1 + (\phi_x)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{k}}}(dx^2 + dy^2).$$

It should be noted that the equation (58) is a well-known *minimal surface equation* for $\kappa = 1$. For $\kappa = -1$, the resulting equation is known to physicists as the *Born-Infeld equation*. Geometrically it is in fact the equation of timelike minimal surfaces in $\mathbb{E}^3_1$. To see this, the area functional of timelike surfaces $\phi(t,x)$ is given by

$$A = \int \sqrt{1 + (\phi_x)^2 - (\phi_t)^2}dt \wedge dx. \tag{62}$$

If we denote the integrand by $f(\phi, \phi_t, \phi_x)$, then the Euler-Lagrange equation for this action functional is

$$\frac{\partial}{\partial t}\frac{\partial f}{\partial \phi_t} + \frac{\partial}{\partial x}\frac{\partial f}{\partial \phi_x} = 0,$$

which is equivalent to the Born-Infeld equation. For $\kappa = -1$, (61) is written as

$$ds^2 = \frac{-1 + (\phi_t)^2}{\sqrt{-(\phi_t)^2 + (\phi_x)^2 + 1}}(dt^2 + dz^2) +$$
(63)
$$2\frac{\phi_t\phi_x}{\sqrt{-(\phi_t)^2 + (\phi_x)^2 + 1}}(dtdx + dydz) + \frac{1 + (\phi_x)^2}{\sqrt{-(\phi_t)^2 + (\phi_x)^2 + 1}}$$
$$(dx^2 + dy^2).$$

Hence we see that there is explicitly a correspondence between timelike minimal surfaces in $\mathbb{E}^3_1$ and a class of anti-self-dual gravitational instantons described by the metric in (63).

An interesting revelation is that there is a correspondence between flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$ and a class of anti-self-dual gravitational instantons described by the Kähler metric in (55). To see this, let $\varphi : \mathbb{D}(t,x) \longrightarrow \mathbb{H}^3_1(-1)$ be a flat Lorentz surface with a pair of Lorentz holomorphic and Lorentz anti-holomorphic data (f,g) given by

$$f = -\frac{2Q}{H+1},\ g = -\frac{2R}{H+1} \tag{64}$$

as stated in Theorem 5. Then there exists a function $u(t,x)$ such that

$$u_{tt} = \ell,\ u_{tx} = \mathfrak{m},\ u_{xx} = \mathfrak{n},$$

and

$$u_{tt}u_{xx} - u_{tx}^2 = -1$$

as discussed in Section 2. The geometric quantities Q, R, H can be written in terms of $\ell, \mathfrak{m}, \mathfrak{n}$ as

$$\begin{aligned} Q &= \frac{1}{4}(\ell + \mathfrak{n} + 2\mathfrak{m}), \\ R &= \frac{1}{4}(\ell + \mathfrak{n} - 2\mathfrak{m}), \\ H &= \frac{1}{2}(-\ell + \mathfrak{n}). \end{aligned} \tag{65}$$

It follows from (64) and (65) that

$$\begin{aligned} u_{tt} &= -\frac{(1+f)(1-g)}{1-fg}, \\ u_{tx} &= -\frac{f-g}{1-fg}, \\ u_{xx} &= \frac{(1-f)(1-g)}{1-fg}. \end{aligned}$$

Therefore the Kähler metric (55) is written as

$$\begin{aligned} ds^2 = &-\frac{(1+f)(1-g)}{1-fg}(dt^2 + dz^2) - 2\frac{f-g}{1-fg}(dtdx + dydz) + \\ &\frac{(1-f)(1-g)}{1-fg}(dx^2 + dy^2). \end{aligned} \tag{66}$$

Physically $f = f(t+x)$ and $g = g(t-x)$ describe a left-moving and a right-moving traveling waves in the Minkowski plane $\mathbb{E}^2_1$.

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