Harmonic Motion in Hyperbolic Plane

Dr. Sung Lee

School of Mathematics and Natural Sciences, University of Southern Mississippi

June 4, 2021

Outline

Hooke's Law

- A force exerted by an elastic cord or by a spring obeys Hooke's law $F = -kx$ where x is the displacement from the equilibrium position.

Hooke's Law

- A force exerted by an elastic cord or by a spring obeys Hooke's law $F = -kx$ where x is the displacement from the equilibrium position.

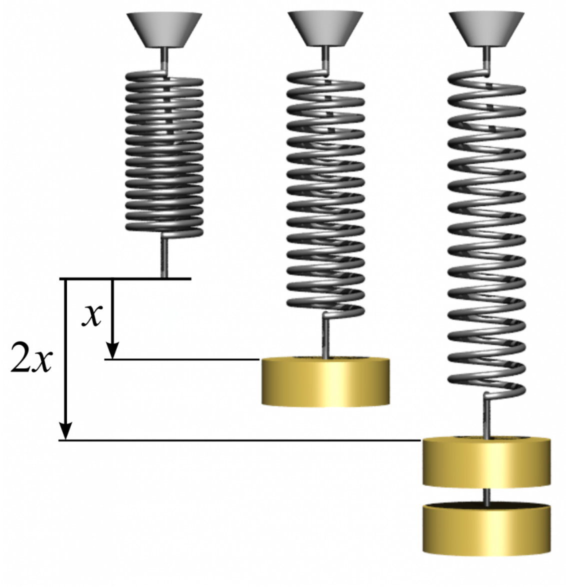

Figure: Hooke's Law

The Differential Equation of Harmonic Motion

- From Newton's second law of motion $F = ma = m\ddot{x}$, the Hooke's law can be written as a *second order linear differential equation*
$$m\ddot{x} + kx = 0$$

The Differential Equation of Harmonic Motion

- From Newton's second law of motion $F = ma = m\ddot{x}$, the Hooke's law can be written as a *second order linear differential equation*
$$m\ddot{x} + kx = 0$$
- To study the motion, we must solve this equation. How do we do that?

Fun Stuff: Solving the Equation of Harmonic Motion

- Notice that the equation of harmonic motion can be viewed approximately as

$$\ddot{x} \sim -x$$

Fun Stuff: Solving the Equation of Harmonic Motion

- Notice that the equation of harmonic motion can be viewed approximately as
$$\ddot{x} \sim -x$$
- This hints us that a solution may be of the form $x(t) = e^{qt}$

Fun Stuff: Solving the Equation of Harmonic Motion

- Notice that the equation of harmonic motion can be viewed approximately as

$$\ddot{x} \sim -x$$

- This hints us that a solution may be of the form $x(t) = e^{qt}$
- Try to see if the trial solution works. If it does, what should be the value of q?

Harmonic Oscillator

- In fact, the trial solution works if q is chosen to be

$$q = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

Harmonic Oscillator

- In fact, the trial solution works if q is chosen to be

$$q = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

- The solution is then

$$x(t) = e^{\pm i\omega_0 t} = \cos\omega_0 t \pm i\sin\omega_0 t$$

Harmonic Oscillator

- In fact, the trial solution works if q is chosen to be

$$q = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

- The solution is then

$$x(t) = e^{\pm i\omega_0 t} = \cos\omega_0 t \pm i\sin\omega_0 t$$

- This complex solution is not suitable for the physical analysis. It turns out the real part $\cos\omega_0 t$ and the imaginary part $\sin\omega_0 t$ are, respectively, also solutions. (Check it for yourself!)

Harmonic Oscillator

Continued

- The superposition of the two real solutions

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

is a solution. Some fancy math theory (*linear algebra*) tells us that this covers all possible real solutions, we call it the *general solution*.

Harmonic Oscillator

Continued

- The superposition of the two real solutions

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

is a solution. Some fancy math theory (*linear algebra*) tells us that this covers all possible real solutions, we call it the *general solution*.

- Using a trigonometric identity $x(t)$ can be written as

$$x(t) = \sqrt{A^2 + B^2}\cos(\omega_0 t - \theta_0)$$

where $\theta_0 = \tan^{-1}\frac{B}{A}$. Then angle θ_0 is called the *phase*.

Harmonic Motion in Euclidean Plane

- The Hooke's law in two dimensions is given by

$$m\ddot{\mathbf{r}} = -k\mathbf{r}$$

where $\mathbf{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y$ is the position vector.

Harmonic Motion in Euclidean Plane

- The Hooke's law in two dimensions is given by

$$m\ddot{\mathbf{r}} = -k\mathbf{r}$$

  where $\mathbf{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y$ is the position vector.

- In terms of the components $x(t)$ and $y(t)$, the Hooke's law can be written as a system of uncoupled differential equations:

$$\ddot{x} = -\omega_0^2 x$$
$$\ddot{y} = -\omega_0^2 y$$

Harmonic Motion in Euclidean Plane

- The Hooke's law in two dimensions is given by

$$m\ddot{\mathbf{r}} = -k\mathbf{r}$$

  where $\mathbf{r}(t) = x(t)\hat{e}_x + y(t)\hat{e}_y$ is the position vector.

- In terms of the components $x(t)$ and $y(t)$, the Hooke's law can be written as a system of uncoupled differential equations:

$$\ddot{x} = -\omega_0^2 x$$
$$\ddot{y} = -\omega_0^2 y$$

- With a suitable change of coordinates, it can be shown that the trajectory of a particle with mass m in the two dimensional potential $V = \frac{1}{2}kr^2$ where $r = \sqrt{x^2 + y^2}$ is an ellipse.

The Lagrangian

- The *Lagrangian* L is defined to be the difference of kinetic and potential energies of a system

$$L(x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n, t) = T - V = \sum_i \frac{1}{2} m \dot{x}_i^2 - V$$

The Lagrangian

- The *Lagrangian* L is defined to be the difference of kinetic and potential energies of a system

$$L(x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n, t) = T - V = \sum_i \frac{1}{2} m \dot{x}_i^2 - V$$

- *Hamilton's principle* in classical mechanics asserts that the motion of the system from time t_1 to t_2 is such that the time integral (*functional*)

$$\int_{t_1}^{t_2} L(x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n, t) dt$$

has a stationary point (critical point).

The Lagrangian
Continued

- Hamilton's principle is equivalent to the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

The Lagrangian

Continued

- Hamilton's principle is equivalent to the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}-\frac{\partial L}{\partial x_i}=0$$

- *Example*. Suppose that F is a conservative force i.e. $F=-\frac{dV(x)}{dx}$. Then

The Euler-Lagrange equation is

$$\frac{d}{dt}m\dot{x}-\frac{\partial(-V)}{\partial x}=m\ddot{x}-F(x)=0$$

which is simply Newton's second law of motion.

Harmonic Motion in Hyperbolic Plane

- We now consider the same potential $V(x, y) = \frac{1}{2}k(x^2 + y^2)$ in hyperbolic plane.

Harmonic Motion in Hyperbolic Plane

- We now consider the same potential $V(x, y) = \frac{1}{2}k(x^2 + y^2)$ in hyperbolic plane.
- We use the *flat chart model* of hyperbolic plane $\mathbb{R}^2$ with metric $ds^2 = dx^2 + e^{2cx}dy^2$. The advantages of working with the flat chart model are that the resulting equation of harmonic motion is simpler and that it can be easily seen that the Euclidean harmonic motion is the limit of hyperbolic harmonic motion as $c \to 0$.

Harmonic Motion in Hyperbolic Plane

- We now consider the same potential $V(x,y)=\frac{1}{2}k(x^2+y^2)$ in hyperbolic plane.
- We use the *flat chart model* of hyperbolic plane $\mathbb{R}^2$ with metric $ds^2=dx^2+e^{2cx}dy^2$. The advantages of working with the flat chart model are that the resulting equation of harmonic motion is simpler and that it can be easily seen that the Euclidean harmonic motion is the limit of hyperbolic harmonic motion as $c\to 0$.
- In hyperbolic plane, the velocity v of a particle is given by

$$v=\sqrt{\dot{x}^2+e^{2cx}\dot{y}^2}$$

Harmonic Motion in Hyperbolic Plane

Continued

- Consequently, the Lagrangian L for hamonic motion in hyperbolic plane is

$$L(\dot{x}, \dot{y}, x, y) = \frac{1}{2}m(\dot{x}^2 + e^{2cx}\dot{y}^2) - \frac{1}{2}k(x^2 + y^2)$$

Research Project

1. Use the Lagrangian, obtain the equation of harmonic motion in hyperbolic plane. (Euler-Lagrange equation.)
2. Solve the resulting equation. If it cannot be solved analytically, try to solve it numerically.
3. Analyze the solution. What can you tell about the trajectory of a particle with mass m in harmonic potential in hyperbolic plane?
4. Make an animation of hyperbolic harmonic motion.
5. Make an animation of hyperbolic harmonic motion at a fixed time t that approach Euclidean harmonic motion as $c \to 0$.

Questions?