MAT 167 Calculus I TEST 1

1. Evaluate the limit
$$\lim_{x\to -3} \frac{x-8}{4x^2-5x+8}$$

   Solution:
$$\begin{aligned}\lim_{x\to -3} \frac{x-8}{4x^2-5x+8} &= \frac{(-3)-8}{4(-3)^2-5(-3)+8}\\ &= -\frac{11}{59}\end{aligned}$$

2. Evaluate the limit
$$\lim_{t\to 1} \frac{t^2-1}{t^2+4t-5}$$

   Solution:
$$\begin{aligned}\lim_{t\to 1} \frac{t^2-1}{t^2+4t-5} &= \lim_{t\to 1} \frac{(t+1)(t-1)}{(t+5)(t-1)}\\ &= \lim_{t\to 1} \frac{t+1}{t+5}\\ &= \frac{1+1}{1+5} = \frac{1}{3}\end{aligned}$$

3. Evaluate the limit, if it exists. If not, state so and explain why.
$$\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5}$$

Solution:

$$\begin{aligned}\lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5} &= \lim_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}\,\frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}\\ &= \lim_{x\to 5}\frac{x-5}{(x-5)\sqrt{x+4}+3}\\ &= \lim_{x\to 5}\frac{1}{\sqrt{x+4}+3}\\ &= \frac{1}{6}\end{aligned}$$

4. Evaluate $\lim_{t\to 0}\frac{\sin 3t}{\sin 8t}$

   Solution:

$$\begin{aligned}\lim_{t\to 0}\frac{\sin 3t}{\sin 8t} &= \frac{3}{8}\lim_{t\to 0}\frac{\frac{\sin 3t}{3t}}{\frac{\sin 8t}{8t}}\\ &= \frac{3}{8}\end{aligned}$$

   because $\lim_{t\to 0}\frac{\sin 3t}{3t}=1$ and $\lim_{t\to 0}\frac{\sin 8t}{8t}=1$.

5. Find the value of the constant b that makes the following function continous on $(-\infty,\infty)$.

$$f(x)=\begin{cases} 6x-6 & \text{if} \quad x\le 3\\ -8x+b & \text{if} \quad x>3\end{cases}$$

   Solution: If $f(x)$ is continuous at $x=3$, it becomes continuous on $(-\infty,\infty)$. For that to happen, we require that $\lim_{x\to 3}f(x)=f(3)$ i.e. $-8(3)+b=6(3)-6$ and hence we find $b=36$.

6. Evaluate the following limits.

   (a)

$$\lim_{x\to\infty}\frac{\sqrt{4+5x^2}}{7+7x}$$

Solution: Divide the numerator and the denominator by $x = \sqrt{x^2}$ to obtain

$$\lim_{x\to\infty} \frac{\sqrt{4+5x^2}}{7+7x} = \lim_{x\to\infty} \frac{\sqrt{\frac{4}{x^2}+5}}{\frac{7}{x}+7} = \frac{\sqrt{5}}{7}$$

(b)

$$\lim_{x\to-\infty} \frac{\sqrt{6+9x^2}}{8+4x}$$

Solution: Recall that $\sqrt{x^2} = |x|$. Since $x \to -\infty$, $x < 0$ and so $\sqrt{x^2} = -x$. Divide the numerator and the denominator by $-x = \sqrt{x^2}$ to obtain

$$\lim_{x\to-\infty} \frac{\sqrt{4+5x^2}}{7+7x} = \lim_{x\to-\infty} \frac{\sqrt{\frac{4}{x^2}+5}}{-\frac{7}{x}-7} = \frac{\sqrt{5}}{-7} = -\frac{\sqrt{5}}{7}$$

7. Evaluate the following limits:

(a)

$$\lim_{x\to\infty} \frac{11x^3-4x^2-2x}{10-5x-9x^3}$$

Solution: Divide the numerator and the denominator of the function by x^3 to obtain

$$\lim_{x\to\infty} \frac{11x^3-4x^2-2x}{10-5x-9x^3} = \lim_{x\to\infty} \frac{11-\frac{4}{x}-\frac{2}{x^2}}{\frac{10}{x^3}-\frac{5}{x^2}-9} = -\frac{11}{9}$$

(b)

$$\lim_{x\to-\infty} \frac{11x^3-4x^2-2x}{10-5x-9x^3}$$

Solution: Divide the numerator and the denominator of the function by x^3 to obtain

$$\lim_{x\to-\infty} \frac{11x^3 - 4x^2 - 2x}{10 - 5x - 9x^3} = \lim_{x\to-\infty} \frac{11 - \frac{4}{x} - \frac{2}{x^2}}{\frac{10}{x^3} - \frac{5}{x^2} - 9}$$
$$= -\frac{11}{9}$$

8. Evaluate the following limit.

$$\lim_{x\to\infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4}$$

Solution: Dividing the numerator and the denominator of the function by x^5, we have

$$\lim_{x\to\infty} \frac{x + x^3 + x^5}{1 - x^2 + x^4} = \lim_{x\to\infty} \frac{\frac{1}{x^4} + \frac{1}{x^2} + 1}{\frac{1}{x^5} - \frac{1}{x^3} + \frac{1}{x}}$$
$$= \infty$$

9. Evaluate the following limit.

$$\lim_{x\to\infty} \frac{x^4 - 5x^2 + 3}{x^5 + 3x^3}$$

Solution: Dividing the numerator and the denominator of the function by x^5, we have

$$\lim_{x\to\infty} \frac{x^4 - 5x^2 + 3}{x^5 + 3x^3} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{5}{x^3} + \frac{3}{x^5}}{1 + \frac{3}{x^2}}$$
$$= 0$$

10. (a) Use the definition of the derivative to compute the derivative of $f(x) = 1 - x^2$ at $x = 3$.

Solution:

$$
\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\
&= \lim_{h\to 0} \frac{1-2(3+h)^2-(1-2(3)^2)}{h} \\
&= \lim_{h\to 0} \frac{1-2(9+6h+h^2)+17}{h} \\
&= \lim_{h\to 0} \frac{-17-12h-2h^2+17}{h} \\
&= \lim_{h\to 0} \frac{-12h-2h^2}{h} \\
&= \lim_{h\to 0}(-12-2h) \\
&= -12
\end{aligned}
$$

(b) Use the definition of the derivative to compute the derivative of $f(x) = 1 - x^2$ at an arbitrary point x.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{1-2(x+h)^2-(1-2x^2)}{h} \\
&= \lim_{h\to 0} \frac{1-2(x^2+2hx+h^2)-1+2x^2}{h} \\
&= \lim_{h\to 0} \frac{-4hx-2h^2}{h} \\
&= \lim_{h\to 0}(-4x-2h) \\
&= -4x
\end{aligned}
$$

(c) Find the slope of tangent line to the graph $y = f(x)$ when $x = 2$.

Solution: The slope of tangent line is

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{9-(2+h)^2-5}{h} \\ &= \lim_{h\to 0} \frac{9-(4+4h+h^2)-5}{h} \\ &= \lim_{h\to 0}(-4-h) \\ &= -4 \end{aligned}$$

(d) Find an equation for the tangent to the curve $f(x) = 1-2x^2$ at $x = 5$.

Solution: The equation of tangent to $y = f(x)$ at $x = a$ is

$$y - f(a) = f'(a)(x-a)$$

Hence we have

$$y - (-49) = -20(x-5)$$

i.e. $y = -20x + 149$.

11. The limit below represents a derivative $f'(a)$. Find $f(x)$ and a.

$$\lim_{h\to 0} \frac{(5+h)^3 - 125}{h}$$

Solution: Comparing the given limit with $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, we find

$$f(x) = x^3,\ a = 5$$