MAT 167 Calculus I TEST 2

1. For which value of x does the graph of $f(x) = x^3 + 3x^2 + 3x + 2$ have a horizontal tangent?

   Solution: The graph of $f(x)$ has a horizontal tangent when $f'(x) = 0$. $f'(x) = 3x^2 + 6x + 3$ and set $3x^2 + 6x + 3 = 0$ which is equivalent to $x^2 + 2x + 1 = (x+1)^2 = 0$. The equation has only one solution $x = -1$ and therefore, the graph of $f(x)$ has a horizontal tangent at $x = -1$.

2. Differentiate the following function:

   $$f(t) = \sqrt{t} - \frac{1}{\sqrt{t}}$$

   Solution: $f(t)$ can be written as $f(t) = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$. Use the Power Rule $(x^n)' = nx^{n-1}$ to obtain

   $$f'(t) = \frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{3}{2}} = \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}}$$

3. Differentiate $f(x) = \frac{x^3 - 5x + 3}{x^3}$.

   Solution: $f(x)$ can be written as

   $$f(x) = 1 - \frac{5}{x^2} + \frac{3}{x^3} = 1 - 5x^{-2} + 3x^{-3}$$

   Hence,

   $$f'(x) = 10x^{-3} - 9x^{-4} = \frac{10}{x^3} - \frac{9}{x^4}$$

4. If $f(x) = \sqrt{x}\sin x$, find $f'(x)$.

Solution: Use the Product Rule to find

$$f'(x) = (\sqrt{x})' \sin x + \sqrt{x}(\sin x)'$$
$$= \frac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$$

5. Let $f(t) = (t^2 + 6t + 4)(5t^2 + 5)$.

   (a) Find $f'(t)$.
   Solution: Use the Product Rule to obtain

$$f'(t) = (t^2 + 6t + 4)'(5t^2 + 5) + (t^2 + 6t + 4)(5t^2 + 5)'$$
$$= (2t + 6)(5t^2 + 5) + (t^2 + 6t + 4)(10t)$$
$$= 10(2t^3 + 9t^2 + 5t + 3)$$

   Note: For WebWork, the last step is not required.

   (b) Find $f'(2)$.
   Solution: $f'(2) = 10(2(2)^3 + 9(2)^2 + 5(2) + 3) = 650$

6. Let $f(x) = \frac{3x^2-8x+4}{3x^2+6x+4}$. Evaluate $f'(x)$ at $x = 4$.
   Solution: Use the Quotient Rule to obtain

$$f'(x) = \frac{(3x^2 - 8x + 4)'(3x^2 + 6x + 4) - (3x^2 - 8x + 4)(3x^2 + 6x + 4)'}{(3x^2 + 6x + 4)^2}$$
$$= \frac{(6x - 8)(3x^2 + 6x + 4) - (3x^2 - 8x + 4)(6x + 6)}{(3x^2 + 6x + 4)^2}$$
$$= \frac{42x^2 - 56}{(3x^2 + 6x + 4)^2}$$

   (Note: For WebWork, the last step is not required.) Hence,

$$f'(4) = \frac{42(4)^2 - 56}{(3(4)^2 + 6(4) + 4)^2} = \frac{616}{5776} = \frac{77}{722}$$

7. If $f(x) = \frac{2\sin x}{3+\cos x}$, find $f'(x)$.

Solution: Use the Quotient ruel to obtain

$$\begin{aligned} f'(x) &= 2\frac{(\sin x)'(3+\cos x) - \sin x(3+\cos x)'}{(3+\cos x)^2} \\ &= 2\frac{\cos x(3+\cos x) - \sin x(-\sin x)}{(3+\cos x)^2} \\ &= 2\frac{3\cos x + 1}{(3+\cos x)^2} \end{aligned}$$

8. Differentiate $y = \sin 6x$.

   Solution: Let $u = 6x$. Then $y = \sin u$. By the Chain rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\cos u)6 = 6\cos 6x \end{aligned}$$

9. Differentiate $f(t) = \ln(\cos t)$.

   Solution: Let $y = \ln(\cos t)$ and $u = \cos t$. Then $y = \ln u$. By the Chain ruel,

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{du}\frac{du}{dt} \\ &= \frac{1}{u}(-\sin t) \\ &= -\frac{\sin t}{\cos t} \\ &= -\tan t \end{aligned}$$

10. Let $f(x) = \sqrt{5x^2 + 4x + 5}$.

    (a) Find $f'(x)$.
    Solution: Let $y = \sqrt{5x^2+4x+5}$ and $u = 5x^2 + 4x + 5$. Then $y = \sqrt{u}$. By the Chain Rule,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}}(10x+4) \\ &= \frac{5x+2}{\sqrt{5x^2+4x+5}} \end{aligned}$$

(b) Find $f'(3)$.

Solution:

$$f'(3) = \frac{5(3)+2}{\sqrt{5(3)^2+4(3)+5}} = \frac{17}{\sqrt{62}}$$

11. If $83+2f(x)+9x^2(f(x))^3=0$ and $f(-3)=-1$, find $f'(-3)$.

Solution: Differentiate $83+2f(x)+9x^2(f(x))^3=0$ with respect to x to obtain

$$2f'(x)+18x(f(x))^3+27x^2(f(x))^2f'(x)=0$$

Plugging in $x=-3$ with $f(-3)=-1$, we have

$$2f'(-3)+18(-3)(-1)^3+27(-3)^2(-1)^2f'(-3)=245f'(-3)+54=0$$

Hence, $f'(-3)=-\frac{54}{245}$.

12. Find an equaton of the line tangent to the curve defined by $x^4+5xy+y^2=7$ at the point $(1,1)$.

Solution: Use the Implicit Differentiation to obtain

$$4x^3+5y+5xy'+2yy'=0$$

Solve the resulting equation for y'.

$$y'=-\frac{4x^3+5y}{5x+2y}$$

Thus

$$\left[\frac{dy}{dx}\right]_{(1,1)}=-\frac{4(1)^3+5(1)}{5(1)+2(1)}=-\frac{9}{7}$$

The equation of tangent line is

$$y-1=-\frac{9}{7}(x-1)$$

i.e.

$$y=-\frac{9}{7}x+\frac{16}{7}$$

13. Let $f(x) = x^{8x}$. Use logarithmic differentiation to determine the derivative.

Solution: Let $y = x^{8x}$. Then

$$\ln y = \ln x^{8x} = 8x \ln x$$

Differentiating this we obtain

$$\frac{1}{y} y' = 8 \ln x + 8x \left(\frac{1}{x} \right) = 8 \ln x + 8$$

Therefore,

$$y' = y(8 \ln x + 8) = x^{8x}(8 \ln x + 8)$$

14. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 9 mi^2/hr. How rapidly is raidus of the spill increasing when the area is 8 mi^2?

Solution: Differentiating $A = \pi r^2$ with respect to t, we obtain

$$\frac{d}{dt} A = 2\pi r \frac{dr}{dt}$$

Thus

$$\frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt}$$

When $A = 8$ mi^2, $r = \sqrt{\frac{8}{\pi}} = 2\sqrt{\frac{2}{\pi}}$. Hence,

$$\frac{dr}{dt} = \frac{1}{4\pi\sqrt{\frac{2}{\pi}}} 9 = \frac{9}{4\sqrt{2\pi}} \text{ mi/hr}$$