MAT 167 Calculus I TEST 3

1. The raidus of a circular disk is given as 29 cm with a maximal error in measurement of 0.3 cm. Use differentials to estimate the following.

   (a) The maximum error in the calculate area of the disk.
   Solution: Differentiate $A = \pi r^2$ with respect to r to obtain

   $$dA = 2\pi r dr$$

   Since $dr \leq 0.3$, $dA \leq 2\pi(29)(0.3) = 17.4$ cm^2.

   (b) The relative maximum error.
   Solution:

   $$\frac{dA}{A} \leq \frac{2\pi(29)(0.3)}{\pi(29)^2} = \frac{2(0.3)}{29} \approx 0.02069$$

   (c) The percentage error in that case.
   Solution: $0.02069 \times 100 = 2.069\%$

2. Find the linear approximation of $f(x) = \ln x$ at $x = 1$ and use it to estimate ln(1.11).
   Solution: With $a = 1$ and $f'(x) = \frac{1}{x}$,

   $$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= x - 1 \end{aligned}$$

   Hence $\ln(1.11) \approx L(1.11) = 1.11 - 1 = 0.11$.

3. Answer the following questions.

   (a) Find differential of the function $y = (x^2 + 2)^3$.
   Solution: $dy = 6x(x^2 + 2)^2 dx$.

(b) When $x = 2$ and $dx = 0.05$, compute the differential.
Solution: $dy = 6(2)(2^2 + 2)^2(0.05) = 21.6$

4. Find the absolute maximum and the absolute minimum values of the function

$$f(x) = x^4 - 98x^2 + 11, \ -6 \leq x \leq 15$$

Solution: $f'(x) = 4x^3 - 196x = 4x(x^2 - 49)$. Setting $f'(x) = 0$ we find critical points $x = -7, 0, 7$. Now

$$\begin{aligned} f(-7) &= -2390 \\ f(-6) &= -2221 \\ f(0) &= 11 \\ f(7) &= 2390 \\ f(15) &= 28586 \end{aligned}$$

Therefore, the abolute maximum value of $f(x)$ is $f(15) = 28586$ and the absolute minimum value of $f(x)$ is $f(\pm 7) = -2390$.

5. At 2:00 pm a car's speedometer reads 30 mph and at 2:10 pm it reads 35 mph. Use the Mean Value Theorem to find an acceleration the car must achieve.

Solution: According to the Mean Value Theorem the car's acceleration at some point has to be the same as the average rate of change of velocity

$$\frac{35 - 30}{2 + \frac{1}{6} - 2} = \frac{5}{\frac{1}{6}} = 30 \text{ mi/h}^2$$

6. Find all number c that satisfy the conclusion of the Mean Value Theorem for the following function and interval.

$$f(x) = 3x^2 + 2x + 6, \ [-1, 1]$$

Solution: By the Mean Value Theorem there is a number $-1 < c < 1$ such that $\frac{f(1) - f(-1)}{1 - (-1)} = f'(c)$. $f'(c) = 6c + 2$ and $\frac{f(1) - f(-1)}{1 - (-1)} = \frac{3 + 2 + 6 - (3 - 2 + 6)}{2} = 2$. Hence solving $6c + 2 = 2$ we find $c = 0$. Figure 1 illustrates the Mean Value Theorem being satisfied for the given function with the c value we found.

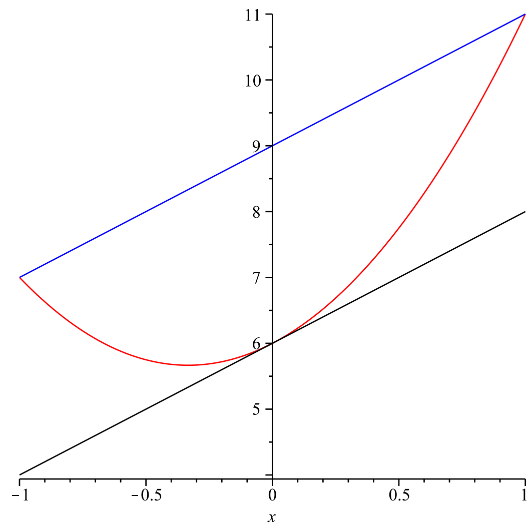

Figure 1: The Mean Value Theorem

7. Let $f(x) = x^3 - 18x^2 + 81x - 4$. Find the open intervals on which f is increasing (decreasing). Then determine the x-coordinates of all relative maxima (minima).

   Solution:

   $$f'(x) = 3x^2 - 36x + 81 = 3(x^2 - 12x + 27) = 3(x-3)(x-9)$$

   Setting $f'(x) > 0$ we find the intervals $(-\infty, 3) \cup (9, \infty)$ on which f is increasing. Setting $f'(x) < 0$ we find the interval $(3, 9)$ on which f is decreasing. Since at the critical point $x = 3$ f changes from increasing to decreasing, f assumes a relative maximum at $x = 3$. Since at the critical point $x = 9$ f changes from decreasing to increasing f assumes a relative minimum at $x = 9$.

8. Let $f(x) = -x^4 - 4x^3 + 6x - 4$. Find the open intervals on which f is concave up(down). Then determine the x-coordinates of all inflection points of f.

Solution:

$$\begin{aligned} f'(x) &= -4x^3 - 12x^2 + 6 \\ f''(x) &= -12x^2 - 24x \\ &= -12x(x+2) \end{aligned}$$

Setting $f''(x) > 0$ we find the interval $(-2, 0)$ on which f is concave up and setting $f''(x) < 0$ we find the intervals $(-\infty, -2) \cup (0, \infty)$ on which f is concave down. Hence the x-coordinates of inflection points are $x = -2, 0$.

9. A box with an open top has vetical sides, a square bottom, and a volume of 256 cubic centimeters. If the box has the least possible surface area, find its dimensions.

   Solution: Let x be the length of the base and y be the height. Then the volume is $x^2y = 256$ and the surface area is $A = x^2 + 4xy$. Solve the volume for y to obtain $y = \frac{256}{x^2}$. Plugging this into A, we write it as a function of a single variable x:

$$A(x) = x^2 + \frac{1024}{x}$$

$$A'(x) = 2x - \frac{1024}{x^2} = 2\left(x - \frac{517}{x^2}\right)$$

   Setting $A'(x) = 0$ we find $x = 8$. Since $A''(x) = 2\left(1 + \frac{1024}{x^3}\right) > 0$ for all $x > 0$, $A(8)$ is the absolute minimum. Therefore the require dimensions are 8 cm × 8 cm × 4 cm.

10. If 1400 square centimeters of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

    Solution: Let x be the length of the base and y be the height. Then the surface area is $x^2 + 4xy = 1400$ and the volume is $V = x^2y$. Solve the area for y to obtain $y = \frac{1400-x^2}{4x}$. Plugging this into V we write V as a function of a single variable x:

$$V(x) = \frac{1}{4}(1400x - x^3)$$

$V'(x) = \frac{1}{4}(1400 - 3x^3)$ and setting it equal to 0 we find $x = \sqrt{\frac{1400}{3}}$. Since $V''(x) = -\frac{3}{2}x < 0$ for all $x > 0$, $V\left(\sqrt{\frac{1400}{3}}\right) \approx 5041$ cm^3 is the absolute maximum.