MAT 168 Calculus II TEST 2

1. Find the area of the region betwen the curves $y = |x|$ and $y = x^2 - 2$.

   Solution: From the picture in Figure 1, the area A is

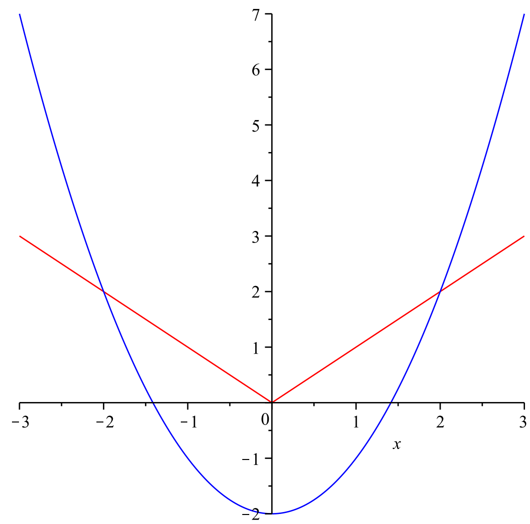

Figure 1: The graphs of $y = |x|$ (in red) and $y = x^2 - 2$ (in blue).

$$
\begin{aligned}
A &= 2\int_0^2 [x - (x^2 - 2)]dx \\
&= 2\left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_0^2 \\
&= \frac{20}{3}
\end{aligned}
$$

2. Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Then find the area of the region.

$$y = 7x, \; y = 5x^2$$

Solution: The equation $5x^2 = 7x$ has solutions $x = 0, \frac{7}{5}$. From the

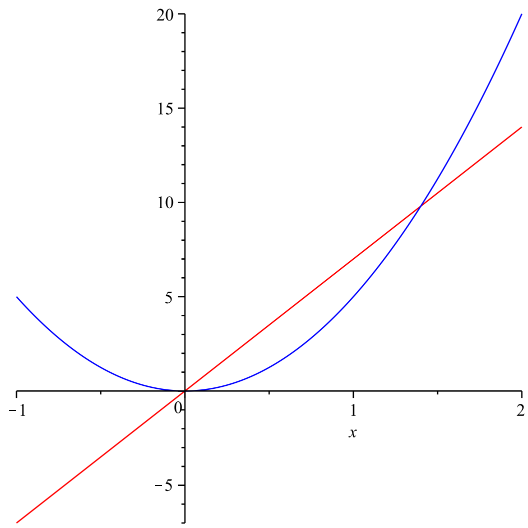

Figure 2: The graphs of $y = 7x$ (in red) and $y = 5x^2$ (in blue).

picture in Figure 2, the area A is

$$\begin{aligned} A &= \int_0^{7/5} [7x - 5x^2]dx \\ &= \left[\frac{7}{2}x^2 - \frac{5}{3}x^3\right]_0^{7/5} \\ &= \frac{7}{2}\left(\frac{7}{5}\right)^2 - \frac{5}{3}\left(\frac{7}{5}\right)^3 \\ &= \frac{1}{6}\frac{7^3}{5^2} \\ &= \frac{343}{150} \end{aligned}$$

3. Find the volume of te solid formed by rotating the region enclosed by

$$x = 0, \ x = 1, \ y = 0, \ y = 4 + x^2$$

about the x-axis.

Solution: The volume V is

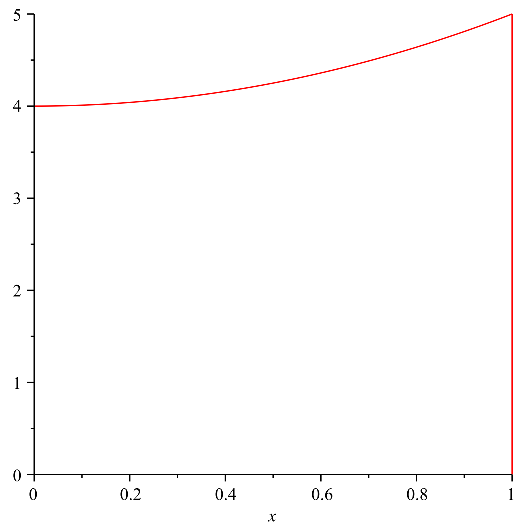

Figure 3: The enclosed region.

$$\begin{aligned} V &= \int_0^1 \pi(4 + x^2)^2 dx \\ &= \pi \int_0^1 (16 + 8x^2 + x^4) dx \\ &= \pi \left[16x + \frac{8}{3}x^3 + \frac{1}{5}x^5 \right]_0^1 \\ &= \frac{283}{15}\pi \end{aligned}$$

4. Find the volume of the solid obtained by rotating the region bounded by

$$x = 5y^2, \ y = 1, \ x = 0$$

about the y-axis.

Solution: The volume V is

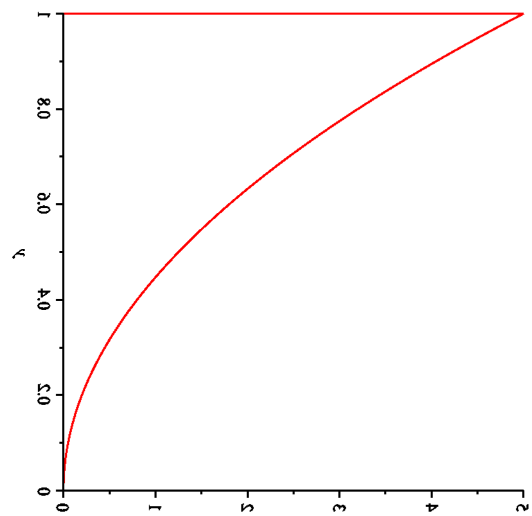

Figure 4: The bounded region.

$$\begin{aligned} V &= \int_0^1 \pi(5y^2)^2 dy \\ &= 25\pi \int_0^1 y^4 dy \\ &= 5\pi[y^5]_0^1 \\ &= 5\pi \end{aligned}$$

5. Using disks or washers, find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $x = y^2$ about the x-axis.

Solution: The volume V is

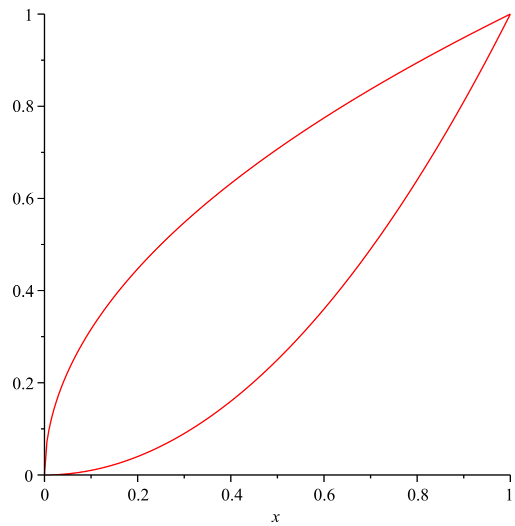

Figure 5: The bounded region.

$$
\begin{aligned}
V &= \int_0^1 \pi[(\sqrt{x})^2 - (x^2)^2]dx \\
&= \pi \int_0^1 [x - x^4]dx \\
&= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \\
&= \frac{3}{10}\pi
\end{aligned}
$$

6. Using disks or washers, find the volume of the solid obtained by rotating the region bounded by the curves $y^2 = x$ and $x = 2y$ about the y-axis.

   Solution: The volume V is

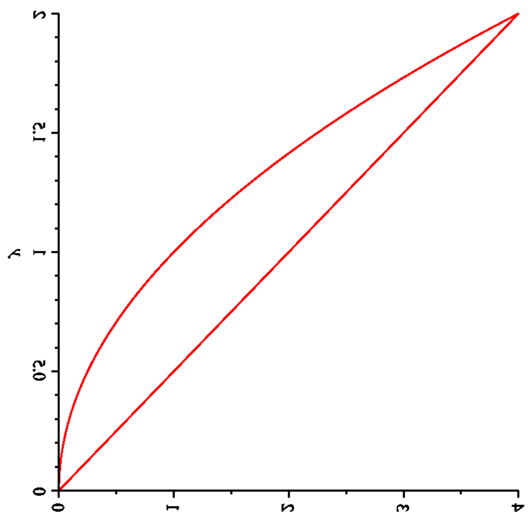

Figure 6: The bounded region.

$$
\begin{aligned}
V &= \int_0^2 \pi[(2y)^2 - (y^2)^2]dy \\
&= \pi \int_0^2 [4y^2 - y^4]dy \\
&= \pi \left[\frac{4}{3}y^3 - \frac{1}{5}y^5 \right]_0^2 \\
&= \frac{64}{15}\pi
\end{aligned}
$$

7. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $x = 4y^2 - y^3$ and $x = 0$ about the x-axis.

   Solution: The volume V is

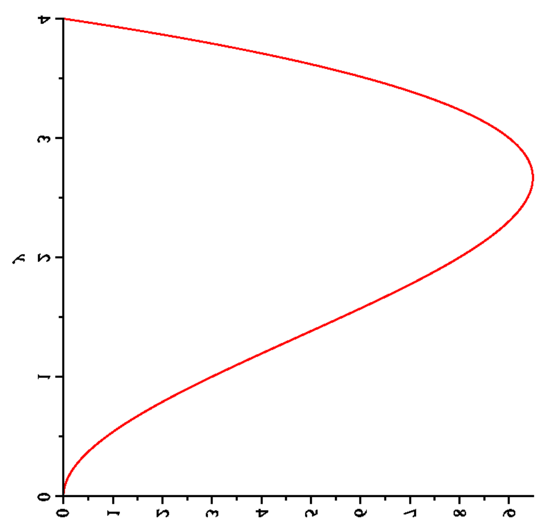

Figure 7: The bounded region.

$$
\begin{aligned}
V &= \int_0^4 2\pi y(4y^2 - y^3)dy \\
&= 2\pi \int_0^4 (4y^3 - y^4)dy \\
&= 2\pi \left[y^4 - \frac{1}{5}y^5 \right]_0^4 \\
&= \frac{2}{5}4^4\pi \\
&= \frac{512}{5}\pi
\end{aligned}
$$

8. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$, $y = 0$, $x = -2$, and $x = -1$ about the y-axis.

   Solution: Note in this case since $-2 \leq x \leq -1$, $x < 0$ and so the

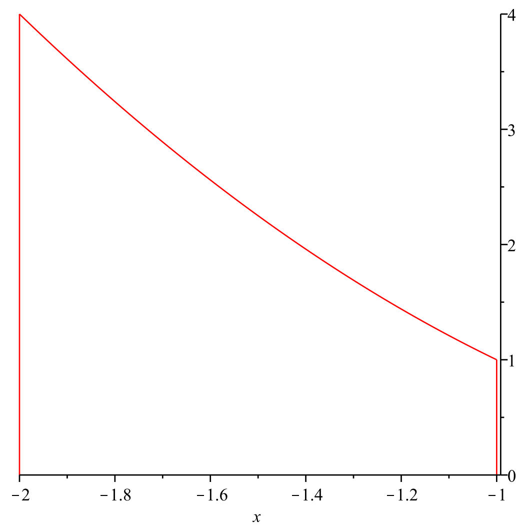

Figure 8: The bounded region.

radius of cylindrical shell at each x is $-x$. Hence the volume is

$$\begin{aligned} V &= \int_{-2}^{-1} 2\pi(-x)(x^2)dx \\ &= -2\pi \int_{-2}^{-1} x^3 dx \\ &= -\frac{1}{2}\pi[x^4]_{-2}^{-1} \\ &= \frac{15}{2}\pi \end{aligned}$$

9. Find the exact length of the curve

$$y = \frac{x^3}{6} + \frac{1}{2x},\ \frac{1}{2} \leq x \leq 1$$

Solution: $y' = \frac{x^2}{2} - \frac{1}{2x^2}$ and

$$\begin{aligned} 1+(y')^2 &= 1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2 \\ &= 1+\left(\frac{x^2}{2}\right)^2 - \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2 \\ &= \left(\frac{x^2}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x^2}\right)^2 \\ &= \left(\frac{x^2}{2}\right)^2 + 2\left(\frac{x^2}{2}\right)\left(\frac{1}{2x^2}\right) + \left(\frac{1}{2x^2}\right)^2 \\ &= \left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2 \end{aligned}$$

Hence, the length of the curve L is

$$\begin{aligned} L &= \int_{1/2}^{1} \sqrt{1+(y')^2}dx \\ &= \int_{1/2}^{1} \left(\frac{x^2}{2}+\frac{1}{2x^2}\right)dx \\ &= \frac{1}{2}\int_{1/2}^{1} \left(x^2+\frac{1}{x^2}\right)dx \\ &= \frac{1}{2}\left[\frac{x^3}{3}-\frac{1}{x}\right]_{1/2}^{1} \\ &= \frac{31}{48} \end{aligned}$$

10. Find the area of the surface otained by rotating the curve

$$y = \sqrt{6x}$$

from $x=0$ to $x=8$ about the x-axis.

Solution: $y' = \frac{3}{\sqrt{6x}}$ and $1+(y')^2 = 1+\frac{3}{2x}$. Hence the surface area

A is

$$\begin{aligned}
A &= \int_0^8 2\pi y\sqrt{1+(y')^2}dx \\
&= 2\pi\int_0^8 \sqrt{6x}\sqrt{1+\frac{3}{2x}}dx \\
&= 2\sqrt{3}\pi\int_0^8 \sqrt{2x+3}dx \\
&= \sqrt{3}\pi\int_3^{19} \sqrt{u}du \text{ (with substitution } u = 2x+3) \\
&= \frac{2\sqrt{3}}{3}\pi[u^{\frac{3}{2}}]_3^{19} \\
&= \frac{2\sqrt{3}}{3}\pi(19^{\frac{3}{2}} - 3^{\frac{3}{2}})
\end{aligned}$$