Doing Quantum Physics with Split-Complex Numbers

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Outline

Path Integral

- In quantum mechanics, the amplitude of a particle to propagate from a point q_I to a point q_F in time T is given by

$$\langle q_F | e^{-\frac{i}{\hbar}\hat{H}T} | q_I \rangle = \int Dq(t) e^{\frac{i}{\hbar}\int_0^T dt L(\dot{q},q)}$$

- $L(\dot{q}, q)$ is the Lagrangian

$$L(\dot{q}, q) = \frac{m}{2}\dot{q}^2 - V(q)$$

- $Dq(t)$ is the Feynman measure given by

$$\int Dq(t) := \lim_{N\to\infty} \left(\frac{-im\hbar}{2\pi\delta t}\right)^{\frac{N}{2}} \left(\prod_{k=1}^{N-1} \int dq_k\right)$$

where $\delta t = \frac{T}{N}$.

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- The path integral is taken over all possible paths from q_I to q_F in spacetime.
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Euclideanisation

- Wick rotation $t \longmapsto it$ turns Minkowski spacetime into Euclidean spacetime.
- Accordnigly, the path integral turns into Euclidean path integral

$$\langle q_F | e^{-\frac{i}{\hbar}\hat{H}T} | q_I \rangle = \int Dq(t) e^{-\frac{1}{\hbar}\int_0^T dt L(\dot{q},q)}$$

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Problems with Euclideanisation

- It is troublesome that path integral cannot be calculated in actual spacetime and that it must be calculated in Euclidean spacetime which is not physical spacetime.
- Most Euclidean solutions are approximations and there is no guarantee that these solutions will be stable when they are brought to Minkowski spacetime.
- Analytic continuation via Wick rotation works when the spacetime is flat. So Euclideanisation will have a problem when the spacetime is curved i.e. gravitation is considered.

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Example

- Define a function $f : (-\infty, 0) \longrightarrow (-\infty, \infty)$ by

$$f(x) = \sum_{n=1}^{\infty} e^{nx}$$
$$= e^{x} + e^{2x} + \cdots$$

- Since $|e^{x}| < 1$ on $(-\infty, 0)$, $f(x)$ converges to

$$f(x) = \frac{1}{e^{-x} - 1}$$

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- The Wick rotated ($x \longmapsto ix$) function

$$g(x) = \sum_{n=1}^{\infty} e^{inx} = \sum_{n=1}^{\infty} [\cos(nx) + i \sin(nx)]$$

does not converge.

- For instance, $g(-2\pi) = \infty$, while $f(-2\pi) = \frac{1}{e^{2\pi}-1}$.

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Why Complex Numbers in Quantum Mechanics?

- Light must be described by electromagnetic waves or by particles (Wave-Particle Duality)
- de Broglie hypothesised that what is true for photons should be valid for any particle.
- A photon can be described by the complex plane wave

$$\psi(\mathbf{r}, t) = A \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$$

  with energy E and momentum vector $\mathbf{p}$ satisfying the equations

$$E = \hbar\omega, \ \mathbf{p} = \hbar\mathbf{k}$$

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Electric-Magnetic Duality

- Maxwell's equations in vacuum are:

$$\nabla \cdot \mathbf{B} = 0, \ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$
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- The transformation

$$\mathbf{B} \mapsto \mathbf{E}, \ \mathbf{E} \mapsto -\mathbf{B}$$

takes the first pair of equations to the second and vice versa. This symmetry is called *Electric-Magnetic Duality*.

- The duality hints that the electric and magnetic fields are part of a unified whole, the *electromagnetic field*.

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Electromagnetic Field as a Complex-Valued Vector Field

- Let us introduce a complex-valued vector field

$$\mathscr{E} = \mathbf{E} + i\mathbf{B}$$

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Plane Wave as Electromagnetic Field

Let $\mathbf{k}$ be a vector in $\mathbb{R}^3$ and let $\omega = |\mathbf{k}|$. Fix $\mathbf{E} \in \mathbb{C}^3$ with $\mathbf{E} \cdot \mathbf{k} = 0$ and $\mathbf{E} \times \mathbf{k} = i\omega\mathbf{E}$. Then the plane wave

$$\mathscr{E}(\mathbf{r}, t) = \mathbf{E} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$$

satisfies the vacuum Maxwell's equations.

The Light Cone and Two-Sphere

- The Light Cone in Minkowski spacetime $\mathbb{R}^{3+1}$ is the hyperquadric

$$\mathbb{N}^3 = \left\{(t,x,y,z) \in \mathbb{R}^{3+1} : t^2 - x^2 - y^2 - z^2 = 0\right\}$$

- Let $\mathbb{N}^3_+$ and $\mathbb{N}^3_-$ denote the future and the past light cones respectively. The multiplicative group $\mathbb{R}^+$ acts on $\mathbb{N}^3_+$ and $\mathbb{N}^3_-$ respectively by scalar multiplication.
- Define $f_\pm : \mathbb{N}^3_\pm \longrightarrow S^2$ by $f_\pm(t,x,y,z) = \left(\frac{x}{t}, \frac{y}{t}, \frac{z}{t}\right)$. Then $f_\pm$ are continuous surjections i.e. identification maps.
- The orbit spaces $\mathbb{N}^3_+/\mathbb{R}^+$ and $\mathbb{N}^3_-/\mathbb{R}^+$ are identified with the two-sphere S^2. The identification is a homeomorphism. It is indeed a diffeomorphism.

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Celestial Sphere and Complex Numbers

- For an observer at the origin (the event), light rays through his eye correspond to null lines through the origin.
- The past null directions constitute the field of vision of the observer which is the two-sphere S^2.
- The two-sphere S^2 is the extended complex plane $\mathbb{C} \cup \{\infty\}$ called the *Riemann sphere*.

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In the beginning, God might have said

"Let there be complex numbers!"

Wave Functions are real?

- In current quantum physics, a wave function itself is not considered as a physical reality but rather a manifestation of something that is both particle and wave.
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Split-Complex Number System

- Let $\mathbb{C}'$ be a real commutative algebra spanned by 1 and j, with multiplication law:

$$1 \cdot j = j \cdot 1 = j, \; j^2 = 1$$

An element of $\mathbb{C}' = 1\mathbb{R} \oplus j\mathbb{R}$ is called a *split-complex number*, a *paracomplex number*, or a *hyperbolic number*.

- $\zeta \in \mathbb{C}'$ is uniquely expressed as $\zeta = x + jy$. The conjugate $\bar{\zeta}$ is defined by $\bar{\zeta} = x - jy$ and the squared modulus $|\zeta|^2$ is defined to be

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- $\mathbb{C}'$ is identified with $\mathbb{R}^{1+1}$.

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Algebraic Representation of $\mathbb{R}^{3+1}$

- The spacetime $\mathbb{R}^{3+1}$ can be identified with a set of 2×2 Hermitian matrices:

$$X = (t, x, y, z) \longleftrightarrow \underline{X} = \begin{pmatrix} t + jz & x + iy \\ x - iy & t - jz \end{pmatrix} = te_0 + xe_1 + ye_2 + jze_3$$

where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \; e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli spin matrices.

- The identification is an isometry:

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In particular, $|X|^2 = \det \underline{X}$.

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Algebraic Representation of $\mathbb{R}^{3+1}$

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- Any four-vector $te_0 + xe_1 + ye_2 + jze_3 \in \mathbb{R}^{3+1}$ can be written as

$$\begin{aligned} te_0 + xe_1 + ye_2 + jze_3 &= (te_0 + jze_3) + (xe_0 + iye_3)e_1 \\ &\longleftrightarrow (t + jz) + (x + iy) \in \mathbb{C}' \oplus \mathbb{C} \end{aligned}$$

- $\mathbb{R}^{3+1} \cong \mathbb{C}' \oplus \mathbb{C}$

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Euler's Formula

- In $\mathbb{C}'$, there is an analogue of the Euler's formula:

$$\exp(j\theta) = \cosh\theta + j\sinh\theta$$

  where $-\infty < \theta < \infty$. The number θ is called a *hyperbolic angle*.

- $\exp(j\theta)$ is a point on the hyperbola $x^2 - y^2 = 1$.
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Split-Complex Plane Wave

- Let us consider a split-complex plane wave $\psi(\mathbf{r},t) = A\exp[j(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, where A is a real number.
- If we assume that the wave is traveling at the speed of light in vacuum, $\psi(\mathbf{r},t)$ satisfies the wave equation

$$-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = 0$$

- The energy operator $\hat{E}$ and the momentum operator $\hat{p}$ are obtained as

$$\hat{E} = -j\hbar\frac{\partial}{\partial t},\ \hat{p} = j\hbar\nabla$$

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- The energy operator $\hat{E}$ and the momentum operator $\hat{p}$ are obtained as

$$\hat{E} = -j\hbar\frac{\partial}{\partial t}, \ \hat{p} = j\hbar\nabla$$

- $\psi(\mathbf{r}, t)$ satisfies the Schrödinger equation

$$-j\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar^2}{2m}\nabla^2\psi$$

Negative Probability?

- Let $\psi^+(\mathbf{r},t) = A\exp[j(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ and $\psi^-(\mathbf{r},t) = Aj\exp[j(\mathbf{k}\cdot\mathbf{r} - \omega t)]$.
- $\psi^-(\mathbf{r}.t)$ also satisfies the Schrödinger equation.
- While $|\psi^+(\mathbf{r},t)|^2 = A^2 > 0$, $|\psi^-(\mathbf{r},t)|^2 = -A^2 < 0$.
- The negative sign may be interpreted as a difference in sign of unit charge between a particle and its antiparticle.
- If $\psi^+(\mathbf{r},t)$ is a plane wave for a particle with charge density $\rho_e^+ = e\overline{\psi^+}\psi^+$, then $\psi^-(\mathbf{r},t)$ may be considered as a plane wave for its antiparticle with charge density $\rho_e^- = e\overline{\psi^-}\psi^-$.

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$\psi^+(\mathbf{r},t)$ and $\psi^-(\mathbf{r},t)$

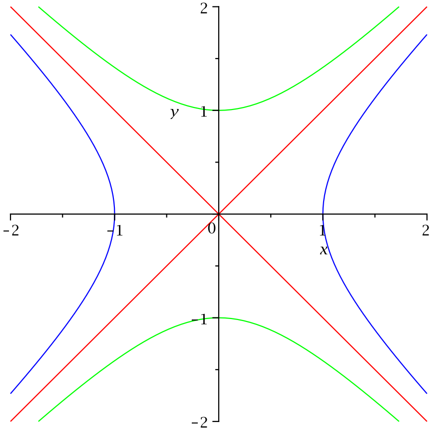

Figure : ψ^+ (in blue) and ψ^- (in green)

Split-Complex Structure and the Charge Conjugation Map

- Define a linear endomorphism $\mathscr{J} : \mathbb{C}' \longrightarrow \mathbb{C}'$ by

$$\mathscr{J} 1 = j, \ \mathscr{J} j = 1$$

- $\mathscr{J}$ satisfies

$$\mathscr{J}^2 = \mathscr{I}, \ \langle \mathscr{J} \zeta_1, \mathscr{J} \zeta_2 \rangle = -\langle \zeta_1, \zeta_2 \rangle$$

  Thus $\mathscr{J}$ is an anti-isometry. $\mathscr{J}$ is called the *associated split-complex structure* of $\mathbb{C}'$.

- $\mathscr{J}$ may be used to define the charge conjugation map on the *split-complex* Hilbert space $\mathscr{H}$ *over real field* $\mathbb{R}$ of state vectors.

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Two Hilbert Spaces $\mathscr{H}^+$ and $\mathscr{H}^-$

- $\{\psi_n^+(\mathbf{r},t) : \psi_n^+(\mathbf{r},t) = A_n \exp[j(\mathbf{k}_n \cdot \mathbf{r} - \omega_n t)],\ n = 1, 2, \cdots\}$ forms a countable basis for a spilt-complex Hilbert space $\mathscr{H}^+$ over real field $\mathbb{R}$.
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Twin Universes

- Under the interpretation, it appears that antiparticles are living in a different spacetime, $\mathbb{R}^{3+1}(t, x, y, z)$ with metric signature $(-+--)$.
- Big Bang might have created twin (not identical though) universes, one made of matter and the other made of antimatter.
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Path Integral Redux

- The amplitude of a particle to propagate from a point q_I to a point q_F in time T is obtained as

$$\langle q_F | e^{-\frac{j}{\hbar}\hat{H}T} | q_I \rangle = \int Dq(t) e^{\frac{j}{\hbar}\int_0^T dt L(\dot{q},q)}$$

- The Feynman meassure $Dq(t)$ is given by

$$\int Dq(t) := \lim_{N\to\infty} \left(\frac{2\pi m\hbar j}{\delta t}\right)^{\frac{N}{2}} \left(\prod_{k=1}^{N-1} \int dq_k\right)$$

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Questions?

Any Questions?