Surfaces of Revolution in Hyperbolic 3-Space

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Outline

Hyperbolic 3-Space $\mathbb{H}^3(-c^2)$

- Let $\mathbb{R}^{3+1}$ denote the Minkowski spacetime with Lorentzian metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

- Hyperbolic 3-space $\mathbb{H}^3(-c^2)$ is the hyperquadric defined by

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -\frac{1}{c^2}.$$

- $\mathbb{H}^3(-c^2)$ has the constant sectional curvature $-c^2$.

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Pseudospherical Model

- On the chart

$$U = \{(x^0, x^1, x^2, x^3) \in \mathbb{H}^3(-c^2) : x^0 + x^1 > 0\}$$

define

$$t = -\frac{1}{c}\log c(x^0 + x^1),$$
$$x = \frac{x^2}{c(x^0 + x^1)},$$
$$y = \frac{x^3}{c(x^0 + x^1)}.$$

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Pseudospherical Model
Continued

- $\mathbb{R}^3$ with coordinates t, x, y and the metric

$$g_c = (dt)^2 + e^{-2ct}\left\{(dx)^2 + (dy)^2\right\}$$

is called the *pseudospherical model* of hyperbolic 3-space.

- The pseudospherical model is a local chart of $\mathbb{H}^3(-c^2)$, so it is not regarded as one of the standard models of hyperbolic 3-space.

- As $c \to 0$, $(\mathbb{R}^3, g_c)$ flattens out to Euclidean 3-space $\mathbb{E}^3$.

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Pseudospherical Model
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- $(\mathbb{R}^3, g_c)$ is isometric to a solvable Lie group G_c with a left-invariant metric

$$G_c = \left\{ \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & e^{ct} & 0 & x \\ 0 & 0 & e^{ct} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} : (t, x, y) \in \mathbb{R}^3 \right\}.$$

- M. Kokubu studied Weierstraß representation of minimal surfaces in $(\mathbb{R}^3, g_c)$ using the solvable Lie group G_c and its Lie algebra. M. Kokubu, *Weiertrass Representation for Minimal Surfaces in Hyperbolic Space*, Tohoku Math. J. **49**, 367-377 (1997)
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Lawson Correspondence

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- Those corresponding constant mean curvature surfaces satisfy the same Gauß-Codazzi equations, so they share many geometric properties in common.
- There is a one-to-one correspondence between surfaces of constant mean curvature H_h in $\mathbb{H}^3(-c^2)$ and surfaces of constant mean curvature $H_e = \pm\sqrt{H_h^2 - c^2}$ in $\mathbb{E}^3$.

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Conformal Parametric Surfaces in $\mathbb{H}^3(-c^2)$

Definition

A parametric surface $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ is said to be *conformal* if

$$\langle \varphi_u, \varphi_v \rangle = 0, |\varphi_u| = |\varphi_v| = e^{\omega/2},$$

where (u, v) is a local coordinate system in M and $\omega : M \to \mathbb{R}$ is a real-valued function in M.

The induced metric on the conformal parametric surface is given by

$$ds_\varphi^2 = e^\omega \left\{ (du)^2 + (dv)^2 \right\}.$$

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Cross Product in $T_p\mathbb{H}^3(-c^3)$

- $\mathbb{H}^3(-c^2)$ is not a vector space but each tangent space $T_p\mathbb{H}^3(-c^2)$ is, and we can consider cross product on each $T_p\mathbb{H}^3(-c^2)$.
- For $\mathbf{v} = v_1 \left(\frac{\partial}{\partial t}\right)_p + v_2 \left(\frac{\partial}{\partial x}\right)_p + v_3 \left(\frac{\partial}{\partial y}\right)_p$, $\mathbf{w} = w_1 \left(\frac{\partial}{\partial t}\right)_p + w_2 \left(\frac{\partial}{\partial x}\right)_p + w_3 \left(\frac{\partial}{\partial y}\right)_p \in T_p\mathbb{H}^3(-c^2)$, define

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Continued

Definition

The cross product $\mathbf{v} \times \mathbf{w}$ is defined by

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2) \left(\frac{\partial}{\partial t} \right)_p + e^{2ct}(v_3 w_1 - v_1 w_3) \left(\frac{\partial}{\partial x} \right)_p + e^{2ct}(v_1 w_2 - v_2 w_1) \left(\frac{\partial}{\partial y} \right)_p,$$

where $p = (t, x, y) \in \mathbb{H}^3(-c^2)$.

The Mean Curvature of a Conformal Parametric Surface in $\mathbb{H}^3(-c^2)$

If a parametric surface $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ is conformal, the mean curvature H is computed by the formula

$$H = \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)},$$

where

$$E = \langle \varphi_u, \varphi_u \rangle, \; F = \langle \varphi_u, \varphi_v \rangle, \; G = \langle \varphi_v, \varphi_v \rangle$$
$$\ell = \langle \varphi_{uu}, N \rangle, \; \mathfrak{m} = \langle \varphi_{uv}, N \rangle, \; \mathfrak{n} = \langle \varphi_{vv}, N \rangle$$

and $N = \frac{\varphi_u \times \varphi_v}{||\varphi_u \times \varphi_v||}$ is a unit normal vector field on φ.

Rotations in $\mathbb{H}^3(-c^2)$

- Rotations about the t-axis are the only type of Euclidean rotations that can be considered in $\mathbb{H}^3(-c^2)$.
- The rotation of a profile curve $\alpha(u) = (u, h(u), 0)$ in the tx-plane about the t-axis through an angle v:

$$\varphi(u, v) = (u, h(u)\cos v, h(u)\sin v).$$

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Differential Equation of $h(u)$ for Surfaces of Revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$

- The mean curvature H of a conformal surface of revolution in $\mathbb{H}^3(-c^2)$ is computed to be

$$H = \frac{-h''(u) + h(u)}{2e^{-2cu}(h(u))^3}.$$

- By setting $H = c$, we obtain the second order non-linear differential equation of $h(u)$

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Limit Behavior of Surfaces of Revolution with CMC $H = c$ as $c \to 0$

- If $c \to 0$, then the differential equation of $h(u)$ becomes

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which is a harmonic oscillator. Its solution is

$$h(u) = c_1 \cosh u + c_2 \sinh u.$$

- For $c_1 = 1, \ c_2 = 0$, we obtain the catenoid

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Catenoid in $\mathbb{E}^3$

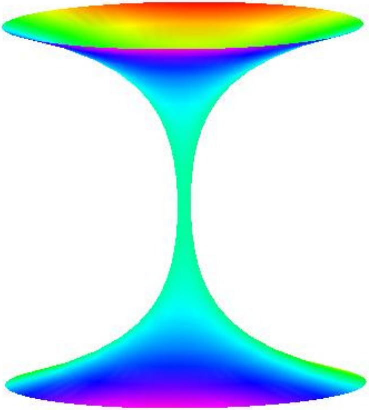


Figure : Catenoid in $\mathbb{E}^3$

Surface of Revolution with CMC $H = 1$ in $\mathbb{H}^3(-1)$

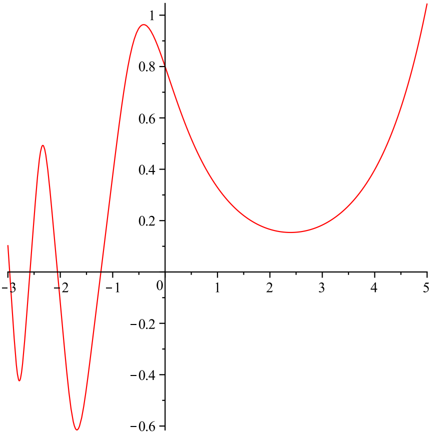
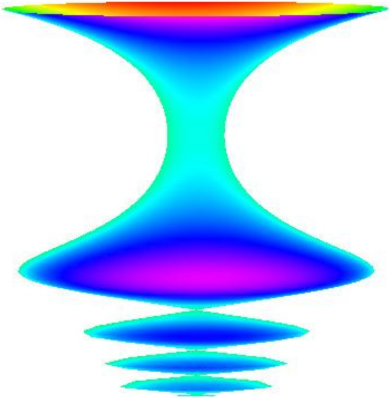


Figure : CMC $H = 1$: Profile Curve

Surface of Revolution with CMC $H = 1$ in $\mathbb{H}^3(-1)$

Continued



Surface of Revolution with CMC $H = \frac{1}{4}$ in $\mathbb{H}^3\left(-\frac{1}{16}\right)$

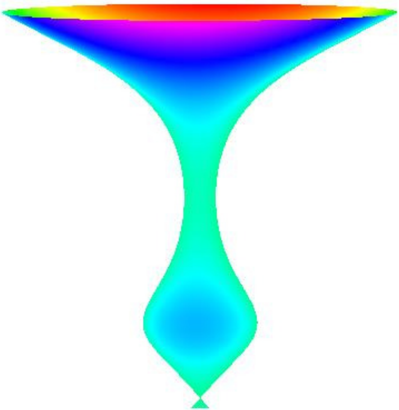


Figure : CMC $H = \frac{1}{4}$: Surface of Revolution

Surface of Revolution with CMC $H = \frac{1}{8}$ in $\mathbb{H}^3\left(-\frac{1}{64}\right)$

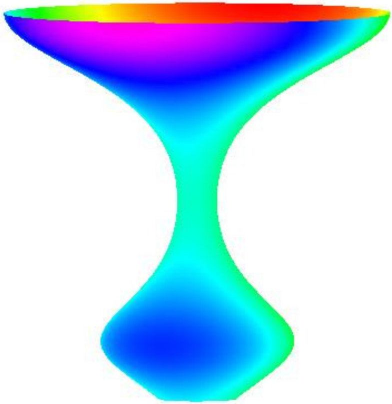


Figure : CMC $H = \frac{1}{8}$: Surface of Revolution

Surface of Revolution with CMC $H = \frac{1}{256}$ in $\mathbb{H}^3\left(-\frac{1}{65536}\right)$

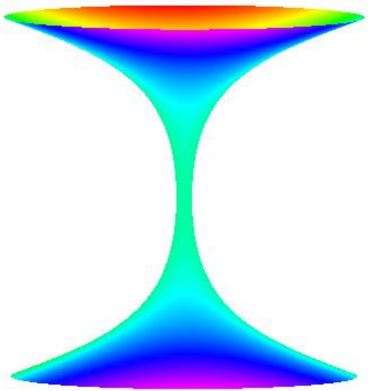


Figure : CMC $H = \frac{1}{256}$: Surface of Revolution

Animations

- Animation of Profile Curves $h(u)$
  `http://www.math.usm.edu/lee/profileanim.gif`
- Animation of Surfaces of Revolution with CMC $H = c$ in $\mathbb{H}^3(-c^2)$
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Harmonic Maps and Minimal Surfaces in $\mathbb{E}^3$

Definition

A smooth map $\varphi : M \longrightarrow \mathbb{E}^3$ is harmonic if it is a critical point of the energy functional

$$E(\varphi) = \frac{1}{2}\int_M ||d\varphi||^2$$

under every compactly supported variation of φ.

- $\varphi : M \longrightarrow \mathbb{E}^3$ is harmonic if and only if $\triangle\varphi = 0$ where $\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is Laplacian.
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Minimal Surfaces in $\mathbb{H}^3(-c^2)$

- In $\mathbb{H}^3(-c^2)$, there is no relationship bewteen minimal surfaces and mean curvature since harmonic map equation is no longer Laplace's equation.
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Construction of Minimal Surface in $\mathbb{H}^3(-c^2)$

- The area functional of $\varphi : M \longrightarrow \mathbb{H}^3(-c^2)$ is

$$J = \int_{t_1}^{t_2} f(x, x_t, t)dt = \int_{t_1}^{t_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dt}\right)^2} dt.$$

- The Euler-Lagrange equation $\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial x_t} = 0$ is

$$\frac{d^2x(t)}{dt^2} - 2\frac{dx(t)}{dt} - x(t) - e^{-2ct}\left(\frac{dx(t)}{dt}\right)^3 = 0.$$

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Minimal Surface of Revolution in $\mathbb{H}^3(-1)$

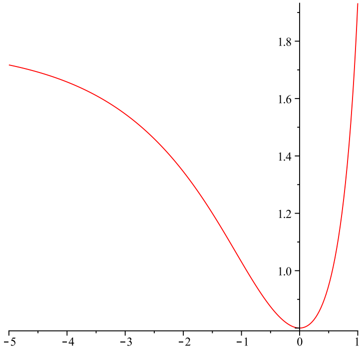

Figure : Minimal Surface of Revolution in $\mathbb{H}^3(-1)$: Profile Curve

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Continued

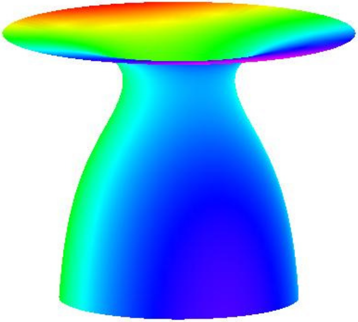

Figure : Minimal Surafce of Revolution in $\mathbb{H}^3(-1)$

Questions?

Any Questions?