The Curvature, the Einstein Equations, and the Black Hole

Sung Lee

University of Southern Mississippi

Physics Seminar, October 11 & Octorber 18, 2019

Outline

The Metric Tensor g_{ij}

- Let (M, g) be an n-dimensional Riemannian or Pseudo-Riemannian manifold. The Riemannian or pseudo-Riemannian metric g can be written locally as

$$g = g_{ij} dx^i \otimes dx^j$$

- The $n \times n$ matrix g_{ij} is called the *metric tensor*.
- Since g_{ij} is a symmetric tensor, it can be diagonalized. Hence WLOG we may assume that $g_{ij} = 0$ if $i \neq j$.

The Christoffel Symbols Γ^k_{ij}

- The Christoffel symbols are associated with the differentiation of vector fields, called the *Levi-Civita connection*.
- The Christoffel symbols are given by

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left\{\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right\},$$

where g^{kl} is the inverse of the metric tensor.

The Curvature

- The Riemann curvature tensor $R^l_{ijk} is given$ by

$$R^l_{ijk} = \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \frac{\partial}{\partial x^k}\Gamma^l_{ij} + \sum_p \left\{ \Gamma^l_{jp}\Gamma^p_{ik} - \Gamma^l_{kp}\Gamma^p_{ij} \right\}$$

- The sectional curvature $K(X,Y)$of $(M,g)$with respect to the plane spanned by $X, Y \in T_pM$ is

$$K_p(X,Y) = g^{ii}R^j_{iji}$$

assuming that $X, Y \in \text{span}\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\}$.

- Ricci cuvature tensor is given by

$$\text{Ric}_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_k R^k_{ikj}$$

We denote $\text{Ric}_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ simply by R_{ij}.

The Curvature

- The Riemann curvature tensor $R^l_{ijk} is given$ by

$$R^l_{ijk} = \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \frac{\partial}{\partial x^k}\Gamma^l_{ij} + \sum_p \left\{ \Gamma^l_{jp}\Gamma^p_{ik} - \Gamma^l_{kp}\Gamma^p_{ij} \right\}$$

- The sectional curvature $K(X,Y)$of $(M,g)$with respect to the plane spanned by $X, Y \in T_pM$ is

$$K_p(X,Y) = g^{ii} R^j_{iji}$$

  assuming that $X, Y \in \mathrm{span}\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\}$.

- Ricci cuvature tensor is given by

$$\mathrm{Ric}_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_k R^k_{ikj}$$

  We denote $\mathrm{Ric}_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ simply by R_{ij}.

The Curvature

- The Riemann curvature tensor $R^l_{ijk} is given$ by

$$R^l_{ijk} = \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \frac{\partial}{\partial x^k}\Gamma^l_{ij} + \sum_p \left\{\Gamma^l_{jp}\Gamma^p_{ik} - \Gamma^l_{kp}\Gamma^p_{ij}\right\}$$

- The sectional curvature $K(X,Y)$of $(M,g)$with respect to the plane spanned by $X, Y \in T_pM$ is

$$K_p(X,Y) = g^{ii}R^j_{iji}$$

assuming that $X, Y \in \mathrm{span}\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right\}$.

- Ricci cuvature tensor is given by

$$\mathrm{Ric}_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \sum_k R^k_{ikj}$$

We denote $\mathrm{Ric}_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ simply by R_{ij}.

The Curvature
Continued

- The scalar curvature $\mathrm{Scal}(p)$ is given by

$$\mathrm{Scal}(p) = \sum_i g^{ii} R_{ii}$$

It is also given in terms of the sectional curvature by

$$\mathrm{Scal}(p) = \sum_{i \neq j} K_p \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right)$$

Maximally Symmetric Spaces

- *Defn.* A Riemannian manifold (M,g) is said to be *maximally symmetric* if (M,g) has constant sectional curvature κ.

- *Thm.* If a Riemannian manifold (M,g) is maximally symmetric, then

$$R_{ii} = \kappa(n-1)g_{ii}$$

whereκ is the sectional curvature of (M,g) and $n = \dim M$.

- *Cor.* If (M,g) has the constant sectional curvature κ, then

$$\mathrm{Scal}(p) = n(n-1)\kappa$$

where $n = \dim M$

Maximally Symmetric Spaces

- *Defn.* A Riemannian manifold (M,g) is said to be *maximally symmetric* if (M,g) has constant sectional curvature κ.
- *Thm.* If a Riemannian manifold (M,g) is maximally symmetric, then
$$R_{ii} = \kappa(n-1)g_{ii}$$
whereκ is the sectional curvature of (M,g) and $n = \dim M$.
- *Cor.* If (M,g) has the constant sectional curvature κ, then
$$\mathrm{Scal}(p) = n(n-1)\kappa$$
where $n = \dim M$

Maximally Symmetric Spaces

- *Defn*. A Riemannian manifold (M,g) is said to be *maximally symmetric* if (M,g) has constant sectional curvature κ.
- *Thm*. If a Riemannian manifold (M,g) is maximally symmetric, then

$$R_{ii} = \kappa(n-1)g_{ii}$$

  whereκ is the sectional curvature of (M,g) and $n = \dim M$.
- *Cor*. If (M,g) has the constant sectional curvature κ, then

$$\mathrm{Scal}(p) = n(n-1)\kappa$$

  where $n = \dim M$

Finding the Fundamental Equations of Gravitation

The fundamental equations of Einstein's theory of gravitation can be obtained by imposing the following requirements.

- The field equations should be independent of coordinatesystems i.e. they should be tensorial.
- Like other field equations, they should be partial differential equations of second order for the componenets g_{ij} of the unknown metric tensor.
- They are a relativistic generalization of the Poisson equation of Newtonian gravitational potential

  $$\nabla^2\phi = 4\pi G\rho$$

  where ρ is the mass density.
- Since the energy-momentum tensor $T_{ij} is$the special relativistic analogue of the mass density, it should be the source of the gravitational field.
- If the space is flat, T_{ij} should vanish.

The Einstein Field Equations

- The equations satisfying all these requirements are given by

$$G_{ij} = 8\pi G T_{ij}$$

where $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$. $G_{ij}$is called the *Einstein tensor*.

- The Einstein Field Equations

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij}$$

were derived independently and almost simultaneously by Einstein and Hilbert (1915)

The Einstein Field Equations

- The equations satisfying all these requirements are given by

$$G_{ij} = 8\pi G T_{ij}$$

where $G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$. $G_{ij}$is called the *Einstein tensor*.

- The Einstein Field Equations

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi G T_{ij}$$

were derived independently and almost simultaneously by Einstein and Hilbert (1915)

Vaccum Field Equations

- Outside the field-producing masses the energy-momentum tensor vanishes and we obtain vaccum field equations

$$R_{ij} - \frac{1}{2} R g_{ij} = 0$$

- For any $i \neq j$, $g_{ij} = 0$ and so $R_{ij} = 0$.
- From $R_{ii} - \frac{1}{2} R g_{ii} = 0$, we have $R - \frac{n}{2} R = \sum_i g^{ii} R_{ii} - \frac{n}{2} R = 0$. Hence $(n-2)R = 0$.
- For $n = 4$, $R = 0$ and hence the vaccum field equations reduce to

$$R_{ij} = 0$$

i.e. vanishing Ricci curvature tensor.

The Schwarzschild Solution

- Consider a static spherically symmetric metric as a solution of the vaccum field equations.

- Ansatz for a static isotropic metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- In addition, we want the solution to be asymptotically flat i.e. it approaches Minkowski flat spacetime if $r \to \infty$.

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

- Although it is a part of the assumptions, a spherically symmetric solution is static i.e. independent of t. This is known as Birkhoff's theorem.

The Schwarzschild Solution

- Consider a static spherically symmetric metric as a solution of the vaccum field equations.
- Ansatz for a static isotropic metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

- In addition, we want the solution to be asymptotically flat i.e. it approaches Minkowski flat spacetime if $r \to \infty$.

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

- Although it is a part of the assumptions, a spherically symmetric solution is static i.e. independent of t. This is known as Birkhoff's theorem.

The Schwarzschild Solution

- Consider a static spherically symmetric metric as a solution of the vaccum field equations.
- Ansatz for a static isotropic metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- In addition, we want the solution to be asymptotically flat i.e. it approaches Minkowski flat spacetime if $r \to \infty$.

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

- Although it is a part of the assumptions, a spherically symmetric solution is static i.e. independent of t. This is known as Birkhoff's theorem.

The Schwarzschild Solution

- Consider a static spherically symmetric metric as a solution of the vaccum field equations.
- Ansatz for a static isotropic metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

- In addition, we want the solution to be asymptotically flat i.e. it approaches Minkowski flat spacetime if $r \to \infty$.

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1$$

- Although it is a part of the assumptions, a spherically symmetric solution is static i.e. independent of t. This is known as Birkhoff's theorem.

Calculations

The nonzero Christoffel symbols are

$$
\begin{aligned}
\Gamma^r_{rr} &= \frac{B'}{2B} \\
\Gamma^r_{tt} &= \frac{A'}{2B} \\
\Gamma^r_{\theta\theta} &= -\frac{r}{B} \\
\Gamma^r_{\phi\phi} &= -\frac{r\sin^2\theta}{B} \\
\Gamma^\theta_{\theta r} &= \Gamma^\phi_{\phi r} = \frac{1}{r} \\
\Gamma^t_{tr} &= \frac{A'}{2A} \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{\phi\theta} &= \cot\theta
\end{aligned}
$$

Calculations
Continued

- The Ricci tensors are

$$
\begin{aligned}
R_{tt} &= \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} \\
R_{rr} &= -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rB} \\
R_{\theta\theta} &= 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \\
R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}
\end{aligned}
$$

- $BR_{tt} + AR_{rr} = \frac{1}{rB}(A'B + B'A)$. Requiring that $(AB)' = (A'B + AB') = 0$ we obtain $AB = \mathrm{const}$. Asymptotic flatness implies that $A(r)B(r) = 1$ i.e $B(r) = \frac{1}{A(r)}$.

Calculations
Continued

- $R_{\theta\theta} = 0$ along with $A'B + AB' = 0$ results in the equation $1 - (Ar)' = 0$ i.e. $(Ar)' = 1$. Hence, $A(r) = 1 + \frac{C}{r}$ for some constant C.
- Newtonian limit:
$$A(r) = -g_{00} = 1 + \frac{2\Phi}{c^2} = 1 - \frac{2MG}{c^2 r}$$
where $\Phi = -\frac{GM}{r}$. Hence, we have $C = -\frac{2MG}{c^2 r}$.
- Schwarzschild metric (1916):
$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Calculations
Continued

- $R_{\theta\theta} = 0$ along with $A'B + AB' = 0$ results in the equation $1 - (Ar)' = 0$ i.e. $(Ar)' = 1$. Hence, $A(r) = 1 + \frac{C}{r}$ for some constant C.
- Newtonian limit:

$$\begin{aligned} A(r) = -g_{00} &= 1 + \frac{2\Phi}{c^2} \\ &= 1 - \frac{2MG}{c^2 r} \end{aligned}$$

where $\Phi = -\frac{GM}{r}$. Hence, we have $C = -\frac{2MG}{c^2 r}$.

- Schwarzschild metric (1916):

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Calculations
Continued

- $R_{\theta\theta} = 0$ along with $A'B + AB' = 0$ results in the equation $1 - (Ar)' = 0$ i.e. $(Ar)' = 1$. Hence, $A(r) = 1 + \frac{C}{r}$ for some constant C.
- Newtonian limit:

$$A(r) = -g_{00} = 1 + \frac{2\Phi}{c^2} = 1 - \frac{2MG}{c^2 r}$$

where $\Phi = -\frac{GM}{r}$. Hence, we have $C = -\frac{2MG}{c^2 r}$.

- Schwarzschild metric (1916):

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$