What is Riemann Hypothesis?

Sungwook Lee

Department of Mathematics, University of Southern Mississippi

Physics Seminar, October 26, 2018

Outline

1. Riemann Zeta-Function

2. Riemann Hypothesis

Dirichlet Series

- Let us consider a Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

  where $s = \sigma + it$ is a complex variable.

- $\zeta(s)$ converges for all complex numbers s with $\sigma = \mathrm{Re}(s) > 1$ and diverges when $\sigma = \mathrm{Re}(s) \leq 1$.

- Bernhard Riemann showed that $\zeta(s)$ can be continued analytically to the punctured complex plane $\mathbb{C} \setminus \{1\}$. The analytic continuation of $\zeta(s)$ is called the *Riemann zeta-function*.

Dirichlet Series

- Let us consider a Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

  where $s = \sigma + it$ is a complex variable.

- $\zeta(s)$ converges for all complex numbers s with $\sigma = \mathrm{Re}(s) > 1$ and diverges when $\sigma = \mathrm{Re}(s) \leq 1$.

- Bernhard Riemann showed that $\zeta(s)$ can be continued analytically to the punctured complex plane $\mathbb{C} \setminus \{1\}$. The analytic continuation of $\zeta(s)$ is called the *Riemann zeta-function.*

Dirichlet Series

- Let us consider a Dirichlet series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

where $s = \sigma + it$ is a complex variable.

- $\zeta(s)$ converges for all complex numbers s with $\sigma = \mathrm{Re}(s) > 1$ and diverges when $\sigma = \mathrm{Re}(s) \leq 1$.

- Bernhard Riemann showed that $\zeta(s)$ can be continued analytically to the punctured complex plane $\mathbb{C} \setminus \{1\}$. The analytic continuation of $\zeta(s)$ is called the *Riemann zeta-function*.

Abel Summation

- Let $\sum_{n=0}^{\infty} a_n$ be an infinite series. Then the limit $\lim_{z\to 1-} \sum_{n=0}^{\infty} a_n z^n$ is called *Abel summation*.

- Let us consider the infinite sequence $a_n = (-1)^n(n+1)$, $n = 0, 1, 2, \cdots$. Then

$$\sum_{n=0}^{\infty} a_n z^n = 1 - 2z + 3z^2 - 4z^3 + \cdots = \frac{1}{(1+z)^2}$$

for $|z| < 1$. So $\lim_{z\to 1-} \sum_{n=0}^{\infty} a_n z^n = \frac{1}{4}$ i.e.

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

Abel Summation

- Let $\sum_{n=0}^{\infty} a_n$ be an infinite series. Then the limit $\lim_{z \to 1-} \sum_{n=0}^{\infty} a_n z^n$ is called *Abel summation*.
- Let us consider the infinite sequence $a_n = (-1)^n(n+1)$, $n = 0, 1, 2, \cdots$. Then

$$\sum_{n=0}^{\infty} a_n z^n = 1 - 2z + 3z^2 - 4z^3 + \cdots = \frac{1}{(1+z)^2}$$

for $|z| < 1$. So $\lim_{z \to 1-} \sum_{n=0}^{\infty} a_n z^n = \frac{1}{4}$ i.e.

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$$

$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots ?$

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots$$
$$2 \cdot 2^{-s}\zeta(s) = 0 + 2 \cdot 2^{-s} + 0 + 2 \cdot 4^{-2} + 0 + 2 \cdot 6^{-s} + \cdots$$

- Subtracting the second identity from the first results

$$(1 - 2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots$$

- $-3\zeta(-1) = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$ and so

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

- The analytic continuation of $\zeta(s)$ evaluated at $s = -1$ is $-\frac{1}{12}$.

$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots ?$

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots$$
$$2 \cdot 2^{-s}\zeta(s) = 0 + 2 \cdot 2^{-s} + 0 + 2 \cdot 4^{-2} + 0 + 2 \cdot 6^{-s} + \cdots$$

- Subtracting the second identity from the first results

$$(1 - 2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots$$

- $-3\zeta(-1) = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$ and so

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

- The analytic continuation of $\zeta(s)$ evaluated at $s = -1$ is $-\frac{1}{12}$.

$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots ?$

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots$$
$$2 \cdot 2^{-s}\zeta(s) = 0 + 2 \cdot 2^{-s} + 0 + 2 \cdot 4^{-2} + 0 + 2 \cdot 6^{-s} + \cdots$$

- Subtracting the second identity from the first results

$$(1 - 2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots$$

- $-3\zeta(-1) = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$ and so

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

- The analytic continuation of $\zeta(s)$ evaluated at $s = -1$ is $-\frac{1}{12}$.

$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots ?$

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots$$
$$2 \cdot 2^{-s}\zeta(s) = 0 + 2 \cdot 2^{-s} + 0 + 2 \cdot 4^{-2} + 0 + 2 \cdot 6^{-s} + \cdots$$

- Subtracting the second identity from the first results

$$(1 - 2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots$$

- $-3\zeta(-1) = 1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$ and so

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

- The analytic continuation of $\zeta(s)$ evaluated at $s = -1$ is $-\frac{1}{12}$.

Riemann's functional equation

- The Riemann zeta-function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

- $\zeta(0) = \frac{1}{\pi} \lim_{s\to 0} \left(\frac{\pi s}{2} - \frac{\pi^3 s^3}{48} + \cdots\right)\left(-\frac{1}{2} + \cdots\right) = -\frac{1}{2}$. Hence we obtain the sum $1 + 1 + 1 + \cdots = -\frac{1}{2}$.
- For $s = -2, -4, -6, \cdots$, $\zeta(s) = 0$. $s = -2, -4, -6, \cdots$ are called the *trivial zeros* of $\zeta(s)$.
- There are also nontrivial zeros of $\zeta(s)$.

Riemann's functional equation

- The Riemann zeta-function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

- $\zeta(0) = \frac{1}{\pi} \lim_{s\to 0} \left(\frac{\pi s}{2} - \frac{\pi^3 s^3}{48} + \cdots\right)\left(-\frac{1}{2} + \cdots\right) = -\frac{1}{2}$. Hence we obtain the sum $1 + 1 + 1 + \cdots = -\frac{1}{2}$.

- For $s = -2, -4, -6, \cdots$, $\zeta(s) = 0$. $s = -2, -4, -6, \cdots$ are called the *trivial zeros* of $\zeta(s)$.

- There are also nontrivial zeros of $\zeta(s)$.

Riemann's functional equation

- The Riemann zeta-function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

- $\zeta(0) = \frac{1}{\pi} \lim_{s\to 0}\left(\frac{\pi s}{2} - \frac{\pi^3 s^3}{48} + \cdots\right)\left(-\frac{1}{2} + \cdots\right) = -\frac{1}{2}$. Hence we obtain the sum $1 + 1 + 1 + \cdots = -\frac{1}{2}$.
- For $s = -2, -4, -6, \cdots$, $\zeta(s) = 0$. $s = -2, -4, -6, \cdots$ are called the *trivial zeros* of $\zeta(s)$.
- There are also nontrivial zeros of $\zeta(s)$.

Riemann's functional equation

- The Riemann zeta-function $\zeta(s)$ satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

- $\zeta(0) = \frac{1}{\pi} \lim_{s\to 0} \left(\frac{\pi s}{2} - \frac{\pi^3 s^3}{48} + \cdots\right) \left(-\frac{1}{2} + \cdots\right) = -\frac{1}{2}$. Hence we obtain the sum $1 + 1 + 1 + \cdots = -\frac{1}{2}$.

- For $s = -2, -4, -6, \cdots$, $\zeta(s) = 0$. $s = -2, -4, -6, \cdots$ are called the *trivial zeros* of $\zeta(s)$.

- There are also nontrivial zeros of $\zeta(s)$.

Riemann Hypothesis

- There is a famous conjecture called the *Riemann Hypothesis* which asserts that
- *The nontrivial zeros of* $\zeta(s)$ *all have the real part* $\mathrm{Re}(s) = \frac{1}{2}$.
- This conjecture has not been resolved. It is a part of Hilbert's eighth problem along with the Golbach conjecture in the *Hilbert's 23 unsolved problems* and is also one of the Clay Mathematics Institute *Millennium Prize Problems*.
- Arguably the Riemann hypothesis is the greatest unsolved problem in mathematics. David Hilbert said *"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"*

Riemann Hypothesis

- There is a famous conjecture called the *Riemann Hypothesis* which asserts that
- *The nontrivial zeros of* $\zeta(s)$ *all have the real part* $\mathrm{Re}(s) = \frac{1}{2}$.
- This conjecture has not been resolved. It is a part of Hilbert's eighth problem along with the Golbach conjecture in the *Hilbert's 23 unsolved problems* and is also one of the Clay Mathematics Institute *Millennium Prize Problems*.
- Arguably the Riemann hypothesis is the greatest unsolved problem in mathematics. David Hilbert said *"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"*

Riemann Hypothesis

- There is a famous conjecture called the *Riemann Hypothesis* which asserts that
- *The nontrivial zeros of* $\zeta(s)$ *all have the real part* $\mathrm{Re}(s) = \frac{1}{2}$.
- This conjecture has not been resolved. It is a part of Hilbert's eighth problem along with the Golbach conjecture in the *Hilbert's 23 unsolved problems* and is also one of the Clay Mathematics Institute *Millennium Prize Problems*.
- Arguably the Riemann hypothesis is the greatest unsolved problem in mathematics. David Hilbert said *"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"*

Riemann Hypothesis

- There is a famous conjecture called the *Riemann Hypothesis* which asserts that
- *The nontrivial zeros of* $\zeta(s)$ *all have the real part* $\mathrm{Re}(s) = \frac{1}{2}$.
- This conjecture has not been resolved. It is a part of Hilbert's eighth problem along with the Golbach conjecture in the *Hilbert's 23 unsolved problems* and is also one of the Clay Mathematics Institute *Millennium Prize Problems*.
- Arguably the Riemann hypothesis is the greatest unsolved problem in mathematics. David Hilbert said *"If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"*

Riemann Hypothesis
Continued

- There is computational evidence that supports the Riemann hypothesis. Notably, Alan Turing found 1,104 nontrivial zeros in 1953. The latest one I know of is 10 trillion (10,000,000,000,000) nontrivial zeros found by X. Gourdon in 2004.

Hilbert-Pólya Conjecture

- Hilbert-Pólya conjecture: Zeros of the Riemann zeta-function $\zeta(s)$ are given by $\frac{1}{2} + i\gamma_j$ where the γ_j are the eigenvalues of a Hermitian Hamiltonian.
- Mathematician Hugh Montgometry discovered the *pair correlation function*

$$1 - \frac{\sin^2(\pi x)}{(\pi x)^2}$$

for zeros of the Riemann zeta-function. Physicist Freeman Dyson recognized that it is also the pair correlation function for eigenvalues in a Gaussian unitary ensemble. Dyson also recognized that zeros of the Riemann zeta-function would form a quasicrystal if the Riemann hypothesis were true. He suggested trying to prove the Riemann hypothesis by classifying 1-dimensional quasicrystals.

Hilbert-Pólya Conjecture

- Hilbert-Pólya conjecture: Zeros of the Riemann zeta-function $\zeta(s)$ are given by $\frac{1}{2} + i\gamma_j$ where the γ_j are the eigenvalues of a Hermitian Hamiltonian.

- Mathematician Hugh Montgomery discovered the *pair correlation function*
$$1 - \frac{\sin^2(\pi x)}{(\pi x)^2}$$
for zeros of the Riemann zeta-function. Physicist Freeman Dyson recognized that it is also the pair correlation function for eigenvalues in a Gaussian unitary ensemble. Dyson also recognized that zeros of the Riemann zeta-function would form a quasicrystal if the Riemann hypothesis were true. He suggested trying to prove the Riemann hypothesis by classifying 1-dimensional quasicrystals.

Hilbert-Pólya Conjecture
Continued

- Montgomery-Odlyzko Law: *The distribution of spacings between nontrivial zeros of the Riemann zeta function is statistically identical to the distribution of eigenvalue spaces in a Gaussian unitary ensemble.*

Latest attempts

- *Hamiltonian for the Zeros of the Riemann Zeta Function*, Carl M. Bender, Dorje C. Brody, and Markus P. Müller, Phys. Rev. Lett. **118**, 130201, 2017
- Sir Michael Atiyah claimed a proof of the Riemann hypothesis at Heidelberg Laureate Forum on 9/25/2018
- So has the Riemann hypothesis been proved? No not really, not even close.

Latest attempts

- *Hamiltonian for the Zeros of the Riemann Zeta Function*, Carl M. Bender, Dorje C. Brody, and Markus P. Müller, Phys. Rev. Lett. **118**, 130201, 2017
- Sir Michael Atiyah claimed a proof of the Riemann hypothesis at Heidelberg Laureate Forum on 9/25/2018
- So has the Riemann hypothesis been proved? No not really, not even close.

Latest attempts

- *Hamiltonian for the Zeros of the Riemann Zeta Function*, Carl M. Bender, Dorje C. Brody, and Markus P. Müller, Phys. Rev. Lett. **118**, 130201, 2017
- Sir Michael Atiyah claimed a proof of the Riemann hypothesis at Heidelberg Laureate Forum on 9/25/2018
- So has the Riemann hypothesis been proved? No not really, not even close.