On $\mathscr{P}$-Hermitian Quantum Mechanics

Sungwook Lee

Department of Mathematics, University of Southern Mississippi

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Outline

$\mathscr{P}$-Hermtian Matrices

- Let $\mathbb{C}^2$ denote the complex 2-dimensional vector space

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

- For $v, w \in \mathbb{C}^2$, define

$$\begin{aligned} \langle v, w \rangle &= \langle v | \mathscr{P} | w \rangle \\ &= v^\dagger \mathscr{P} w \end{aligned}$$

where $v^\dagger = \bar{v}^t$ and $\mathscr{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\langle \, , \, \rangle$ defines an indefinte Hermitian product on $\mathbb{C}^2$.

- *Defn.* A 2×2 complex matrix H is called *$\mathscr{P}$-Hermitan* if

$$\mathscr{P} H^\dagger \mathscr{P}^{-1} = H$$

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$\mathscr{P}$-Hermitan Matrices
Continued

- If H is $\mathscr{P}$-Hermitian, H can be written as

$$H = \begin{pmatrix} a & b \\ -\bar{b} & d \end{pmatrix}$$

where a and d are real numbers.

Time Evolution

- Let $U(t) = \exp\left(-\frac{i}{\hbar}Ht\right)$. Then $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ is a solution of the Schrödinger equation

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

- $U(t)$ is called the *time-evolution operator*.
- $U(t)$ is said to be *unitary* if it is an isometry i.e. $\langle\psi(t),\psi(t)\rangle = \langle\psi(0),\psi(0)\rangle$ for all t.
- *Thm.* $U(t)$ is unitary if and only if

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Time Evolution
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- *Thm.* $U(t)$ is unitary if and only if H is $\mathscr{P}$-Hermition.
- The set of unitary transformations forms a Lie subgroup $U(1,1)$ of $SL(2,\mathbb{C})$. $U(1,1)$ is called the *pseudo unitary group*.
- If $\mathbb{C}^2$ is considered as a 2-dim indefinite Hermitian manifold, the gauge group of the frame bundle $L\mathbb{C}^2$ is $U(1,1)$.
- A 2×2 complex matrix H is $\mathscr{P}$-Hermitian if and only if $-iH \in u(1,1)$, the Lie algebra of $U(1,1)$.

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Time Evolution
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- If $\mathbb{C}^2$ is orientable, the gauge group of $L\mathbb{C}^2$can be reduced to $SU(1,1)$, the special pseudo unitary group. The Lie algebra $su(1,1)$ of $SU(1,1)$ is the set of elements in $u(1,1)$ that are trace-free. With the additional condition $\mathrm{tr}(H) = 0$, a $\mathscr{P}$-Hermitian hamiltonian H can be written as

$$H = \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix}$$

where a is a real number.

$|\psi|^2$ is not a probability!

- Since $|\psi|^2$ could be positive, negative, or zero, $|\psi|^2$ cannot be interpreted as a probability.
- Instead $|\psi|^2$may be considered as an internal symmetry and that the time evolution operator $U(t)$ is required to preserve the internal symmetry analogously to Lorentz transformations.
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- Define $\langle\ ,\ \rangle_+$ by

$$\begin{aligned}\langle v, w\rangle_+ &= \langle v, \mathscr{P} w\rangle \\ &= \langle v|w\rangle \\ &= v^\dagger w\end{aligned}$$

  Then $\langle\ ,\ \rangle_+$defines a positive definte Hermitan product on $\mathbb{C}^2$.

- Physicists considered $|\psi|_+^2 = \langle\psi, \psi\rangle_+$ as a probability.
- However $|\psi|_+^2$ cannot be a probability as it violates unitarity!

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Example

- Let $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Consider time evolution of $|\psi(t)\rangle = \omega_1(t)|\psi_1\rangle + \omega_2(t)|\psi_2\rangle$ with $H = \begin{pmatrix} 3 & -1+i \\ 1+i & -3 \end{pmatrix}$.
- $|\psi(t)|^2$ is preserved as expected.
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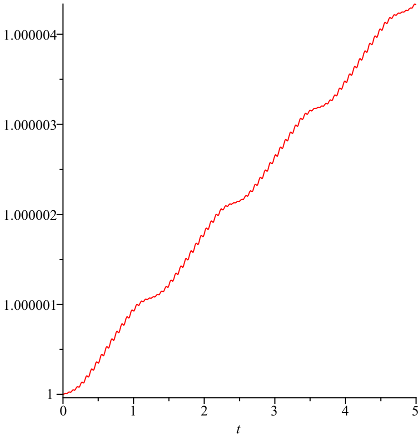

Figure : $|\psi(t)|^2$

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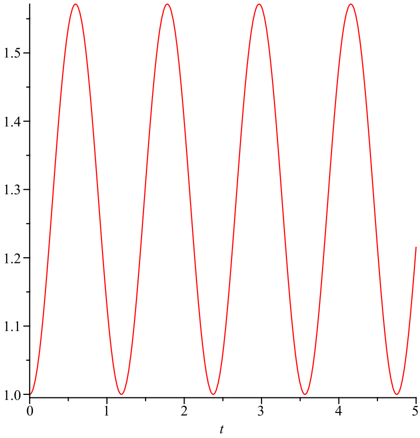

Figure : $|\psi(t)|_{+}^{2}$

The probability of the system being in the state $|\psi_i\rangle$

- The probability of the system being in the state $|\psi_i\rangle$ can be defined in the same way it is defined in the standard quantum mechanics

$$Pr(|\psi_i\rangle) = \frac{|\omega_i|^2}{|\omega_1|^2 + |\omega_2|^2} = \frac{|\langle\psi_i, \psi\rangle|^2}{\sum_{j=1}^{2} |\langle\psi_j, \psi\rangle|^2}$$

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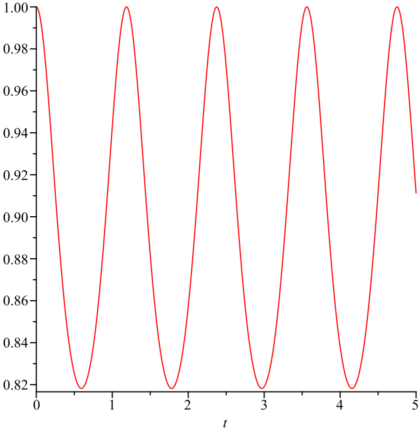

Figure : $Pr(|\psi_1\rangle)$

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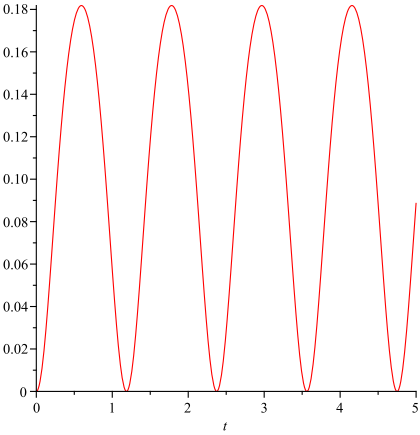

Figure : $Pr(|\psi_2\rangle)$

Spin-Flip
The Rabi Experiment

- Pauli equation in $\mathscr{P}$-Hermitian quantum mechanics

$$\begin{aligned} i\hbar\frac{d|\psi(t)\rangle}{dt} &= -B\cdot M|\psi(t)\rangle \\ &= -\mu B\cdot\sigma|\psi(t)\rangle \end{aligned}$$

where $B = (B_0\cos\omega_0 t, B_0\sin\omega_0 t, B_z)$, $|\psi(t)\rangle = a(t)e^{-i\omega t}|\psi_1\rangle + b(t)e^{i\omega t}|\psi_2\rangle$ ($\omega = -\frac{\mu B_z}{\hbar}$ is the Larmor frequency), and σ_i, $i = 1,2,3$ are the Pauli matrices (in $\mathscr{P}$-Hermtian quantum mechanics) given by

$$\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Spin-Flip
Continued

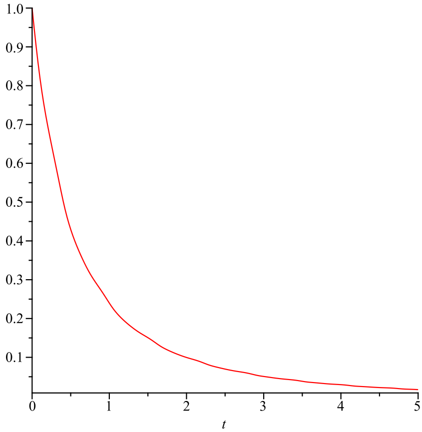

Figure : $|\psi(t)|^2 = |a(t)|^2 - |b(t)|^2$

Spin-Flip
Continued

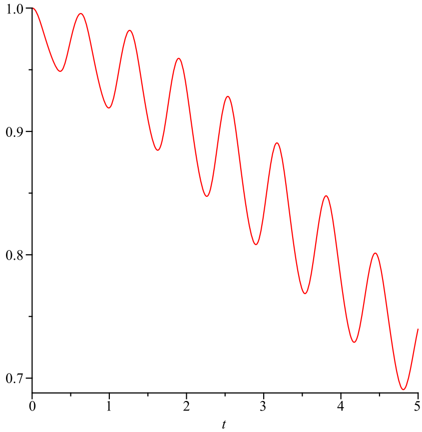

Figure : $Pr(|\psi_1\rangle)$

Spin-Flip
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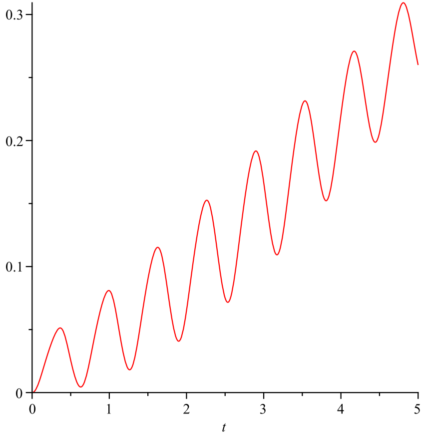

Figure : $Pr(|\psi_2\rangle)$

Symmetry of $\mathscr{P}$-Hermitian Quantum Mechanics

- Recall that the set of unitary transformations is the Lie group $SU(1,1)$. $SU(1,1)$ is the universal cover of the Lorentz group $SO^+(2,1)$.
- $SO^+(2,1)$ is the symmetry group of Minkowski space $\mathbb{R}^{2+1}$.
- No rotational symmetry $SO(3)$ in $\mathscr{P}$-Hermitian quantum mechanics!
- The universal cover $\rho : SU(1,1) \longrightarrow SO^+(2,1)$ is a double cover and $\ker \rho = \mathbb{Z}_2 = \{\pm I\}$, so

$$SU(1,1)/\mathbb{Z}_2 = SO^+(2,1)$$

Mathematically, this defines spin in $\mathscr{P}$-Hermtian quantum mechanics.

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Quantum Angular Momentum in $\mathscr{P}$-Hermitian Quantum Mechanics

- The symmetry of $\mathscr{P}$-Hermitian quantum mechanics indicates that quantum angular momentum would be different from that of the standard quantum mechanics.
- The quantum angular momentum can be derived from the symmetry as

$$L_t = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$
$$L_x = i\hbar\left(y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}\right)$$
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Indefinte Hermitian Product

- Let $\mathscr{P}$ be the *parity operator* i.e. $\mathscr{P}\psi(x,t) = \psi(-x,t)$.
- Define $\langle\ ,\ \rangle$ on the space of state vectors by

$$\begin{aligned}\langle \varphi, \psi \rangle &= \langle \varphi | \mathscr{P} | \psi \rangle \\ &= \int_{-\infty}^{\infty} \overline{\varphi(x)}\psi(-x)dx\end{aligned}$$

Then $\langle\ ,\ \rangle$ is an indefinite Hermitian product on the space of state vectors.

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Adjoint Operators and Self-Adjointness

- Let A be a linear operator on the space of state vectors. Define its *adjoint* to be the operator A^*satisfying

$$\langle \varphi, A\psi \rangle = \langle A^* \varphi, \psi \rangle$$

- Equivalently

$$\int_{-\infty}^{\infty} \bar{\varphi} \mathscr{P}(A\psi) dx = \int_{-\infty}^{\infty} \overline{A^* \varphi} \mathscr{P} \psi$$

- A linear operator A is said to be *self-adjoint* or *$\mathscr{P}$-Hermitian* if $A = A^*$ or equivalently

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Reality of eigenvalues of $\mathscr{P}$-Hermitian operators

- It is important to require that the eigenvalues (energies) of Hamiltonian operators are real.
- *Thm.* Let L be a $\mathscr{P}$-Hermitian operator. Let λ be a nonreal complex eigenvalue of L. Then its associated eigenvector has vanishing squared norm.
- Equivalently, the eigenvalues of a $\mathscr{P}$-Hermitian operator are real provided their associated eigenvectors have nonvanishing squared norm.

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Time Evolution Operator and $\mathscr{P}$-Hermitian Hamiltonian

- Let H be a time independent Hamiltonian. Then the time evolution operator $U(x,t)$ is given by

$$U(x,t) = \exp\left(-\frac{i}{\hbar}Ht\right)$$

- *Thm.* $U(x,t)$ is unitary if and only if $U^* = U^{-1}$. Using this theorem one can prove that:
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- Defn. Let L be a time independent operator. L is said to be $\mathscr{P}\mathscr{T}$-symmetric if $\mathscr{P}\mathscr{T}L(x) = \overline{L(-x)} = L(x)$.
- *Thm.* If $V(x)$ is a potential energy operator which acts on $|\psi(x)\rangle$ by multiplication, then $V(x)$ is $\mathscr{P}$-Hermitian if and only if it is $\mathscr{P}\mathscr{T}$-symmetric.
- *Cor.* Hamiltonian H of the form

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

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$\mathscr{P}$-Hermiticity and $\mathscr{P}\mathscr{T}$-Symmetry

- Defn. Let L be a time independent operator. L is said to be $\mathscr{P}\mathscr{T}$-symmetric if $\mathscr{P}\mathscr{T}L(x) = \overline{L(-x)} = L(x)$.
- *Thm.* If $V(x)$ is a potential energy operator which acts on $|\psi(x)\rangle$ by multiplication, then $V(x)$ is $\mathscr{P}$-Hermitian if and only if it is $\mathscr{P}\mathscr{T}$-symmetric.
- *Cor.* Hamiltonian H of the form

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

is $\mathscr{P}$-Hermitian if and only if it is $\mathscr{P}\mathscr{T}$-symmetric.

A New Class of Hamiltonians

- In mathematics, any complex-valued function $f(x)$ satisfying

$$\overline{f(-x)} = f(x)$$

  is called a *Hermitian function*. It can be shown that the real and imaginary parts of a Hermitian function is an even and an odd functions respectively.

- $\mathscr{P}$-Hermitian potential operators may be complex while standard Hermitian potential operators must be real.

- Examples of $\mathscr{P}$-Hermtian potential operators $V(x)$ includes

$$ix^3,\ ix^5,\ x^2 + ix^3, e^{ix} = \cos x + i \sin x$$

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Time evolution with positive definite Hermitian product

- $\mathscr{P}\mathscr{T}$-symmetric quantum physicists insist that they should use positive definite Hermitian product to study $\mathscr{P}\mathscr{T}$-symmetric quantum mechanics in order to interpret $|\psi|^2$ as a probability. However this leads to a serious problem for them.

- *Thm.* Let H be a $\mathscr{P}$-Hermitian hamiltonian of the form $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$. Then the time evolution operator $U(x,t)$ is unitary with respect to positive definite Hermitian product if and only if $V(x)$ is real.

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A possible connection to Riemann Hypothesis?

- *Riemann Hypothesis*: All nontrivial zeros of the Riemann zeta-function

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

, where $0 < \Re(s) < 1$, lies on the critical line $\frac{1}{2} + it$.

- *Hilbert-Pólya Conjecture*: The imaginary part of nontrivial zeros $\frac{1}{2} + it$ of the Riemann zeta function $\zeta(s)$ are the eigenvalues of a Hermitian Hamiltonian H of a particle of mass m that is moving under the influence of a potential $V(x)$.

- The Hilbert-Pólya Conjecture can be restated in terms of $\mathscr{P}$-Hermitian hamiltonians.
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A Quote

"The universe is not only stranger than we imagine, it is stranger than we can imagine."
J. B. S. Haldane (5 November 1892 - 1 December 1964), a British biologist and a commie.

Questions?

Any Questions?