# Numbers and Quantum Physics

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# Outline

## Wave-Particle Duality

- In the earlier investigations of light, physicists realized light exhibits both the nature of waves and that of particles. (*Wave-particle duality*.)
- In his 1924 Ph.D. thesis, Louis de Broglie proposed an ambitious hypothesis that wave-particle duality is not limited to light but applies to all matter. Furthermore he hypothesized that what is true for photons should be valid for any particle.
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# A Complex Plane Wave Model of an Electron

- It was known that photons can be described by the complex plane wave, called the *de Broglie wave*

$$\psi(\mathbf{r},t) = A\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

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$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

which is called the Schrödinger equation.

- $\psi(\mathbf{r}, t)$ also satisfies the wave equation

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# Path Integral

- In quantum mechanics, the probability amplitude of a particle to propagate from a point $q_I$ to a point $q_F$ in time $T$ is given by

$$\langle q_F | e^{-\frac{i}{\hbar}\hat{H}T} | q_I \rangle = \int Dq(t) e^{\frac{i}{\hbar}\int_0^T dt L(\dot{q},q)}$$

- $L(\dot{q}, q)$ is the Lagrangian

$$L(\dot{q}, q) = \frac{m}{2}\dot{q}^2 - V(q)$$

- $Dq(t)$ is the Feynman measure given by

$$\int Dq(t) := \lim_{N\to\infty} \left(\frac{-im\hbar}{2\pi\delta t}\right)^{\frac{N}{2}} \left(\prod_{k=1}^{N-1} \int dq_k\right)$$

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# Euclideanization

## A Clever But Bizarre Remedy

- *Wick rotation* $t \longmapsto it$ turns Minkowski spacetime into Euclidean spacetime.
- Accordnigly, the path integral turns into Euclidean path integral

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# Problems with Euclideanization

- It is troublesome that path integral cannot be calculated in actual spacetime and that it must be calculated in Euclidean spacetime which is not a physical spacetime.
- Most Euclidean solutions are approximations and there is no guarantee that these solutions will be stable when they are brought to Minkowski spacetime.
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# Complex Numbers Are For Light

- Is it a coincidence that photons are described by complex numbers?
- It appears that complex numbers are meant for light.
- We view the sky as a round sphere, called the *celestial sphere*, and the objects in the sky are projected onto the inner surface of the sphere. This is because we see by our eyes which detect only photons. This can be, in fact, mathematically proven and the 2-sphere can be unwrapped to the complex plane = the Euclidean plane (via stereographic projection).
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- What if our eyes detect electrons instead of photons? What would the sky look like?
- If the surface of the screen upon which the celestial objects are projected has a constant curvature, it may be a pseudosphere.
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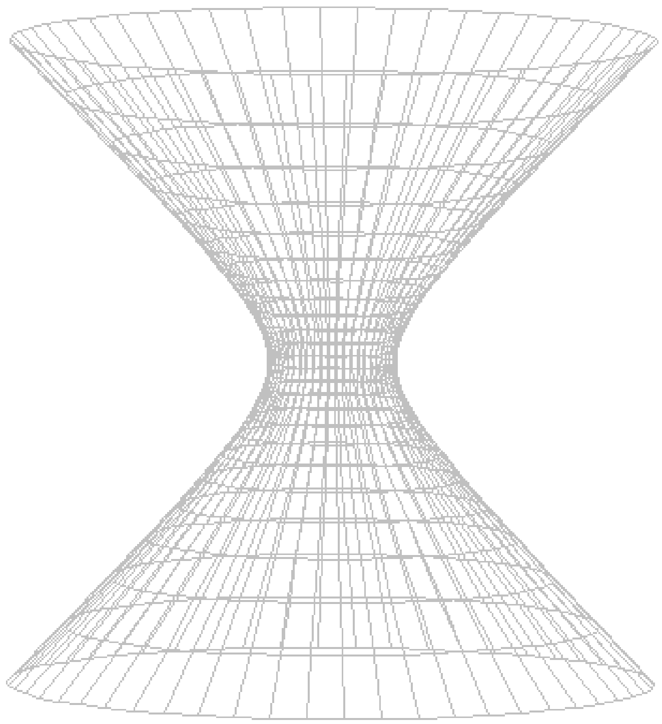

Figure: A Pseudosphere

# Do we have an alternative?

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- Question: Can we possibly formulate quantum mechanics with a different number system?
- The answer appears to be affirmative.

# Split-Complex Number System

- Let $\mathbb{C}'$ be a real commutative algebra spanned by 1 and $j$, with multiplication law:

$$1 \cdot j = j \cdot 1 = j, \; j^2 = 1$$

An element of $\mathbb{C}' = 1\mathbb{R} \oplus j\mathbb{R}$ is called a *split-complex number*, a *paracomplex number*, or a *hyperbolic number*.

- $\zeta \in \mathbb{C}'$ is uniquely expressed as $\zeta = x + jy$. The conjugate $\bar{\zeta}$ is defined by $\bar{\zeta} = x - jy$ and the squared modulus $|\zeta|^2$ is defined to be

$$|\zeta|^2 = \zeta\bar{\zeta} = x^2 - y^2$$

- $\mathbb{C}'$ is identified with $\mathbb{R}^{1+1}$ with metric $dx^2 - dy^2$.

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# Euler's Formula

- In $\mathbb{C}'$, there is an analogue of the Euler's formula:

$$e^{j\rho} = \cosh\rho + j\sinh\rho$$

where $-\infty < \rho < \infty$. The number $\rho$ is called a *hyperbolic angle*.

- $e^{j\rho}$ is a point on the hyperbola $x^2 - y^2 = 1$.
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# Split-Complex Plane Wave

- We may consider the split-complex plane wave $\psi(\mathbf{r}, t) = A\exp[j(\mathbf{k} \cdot \mathbf{r} - \omega t)]$, where $A$ is a real number, as the building blocks of a quantum mechanics model.
- The fronts of the split-complex plane wave are hyperbolas.
- Optical waves propagating from a point source exhibit circular wavefronts like the de Broglie wave. The reason is that the medium through which light travels is typically homogeneous and isotropic.
- Interestingly, a group of material scientists in Spain developed surfaces called *hyperbolic metasurfaces* on which the waves emitted from a point source propagate only in certain directions with open wave fronts.

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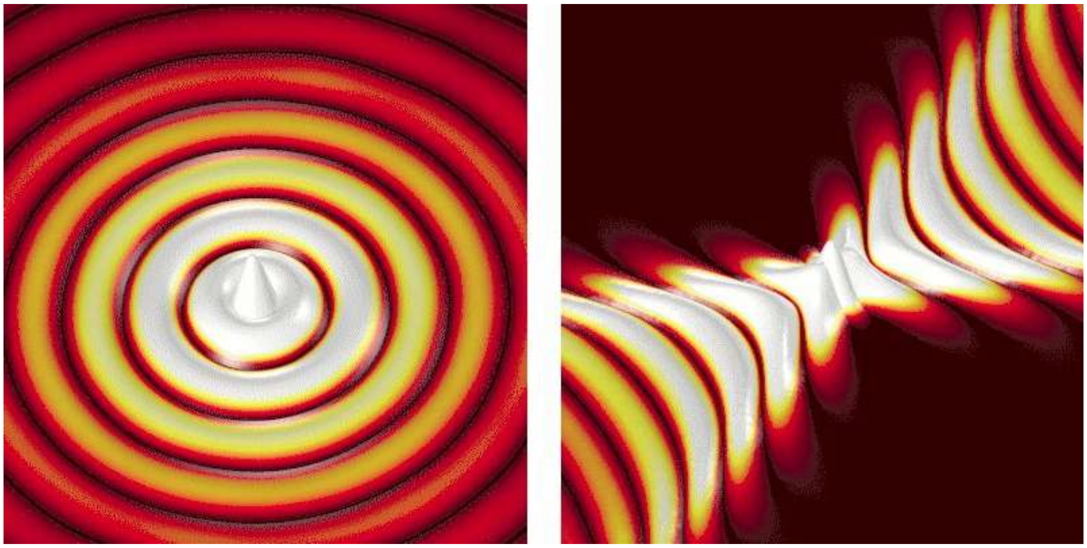

Figure: Circular wave fronts v.s. hyperbolic wave fronts

Source: Peining Li et al., Infrared hyperbolic metasurface based on nanostructured van der Waals materials, Science (2018)

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# The Plane Wave with a Negative Probability?

- The plane wave $j\psi(\mathbf{r},t) = jA\exp[j(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ satisfies the same Schrödinger type equation as $\psi(\mathbf{r},t)$ does but $|j\psi(\mathbf{r},t)|^2 = -A^2 < 0$.
- The probability density being negative is troublesome. Initially this led me to believe that this whole idea is a flop. But then ...
- It turns out that the particle described by $j\psi(\mathbf{r},t)$ does not belong to the same universe where the particle described by $\psi(\mathbf{r},t)$ belongs to. In its own universe $j\psi(\mathbf{r},t)$ has a positive squared norm (probability density). Likewise, $\psi(\mathbf{r},t)$ has a negative squared norm in the other universe.
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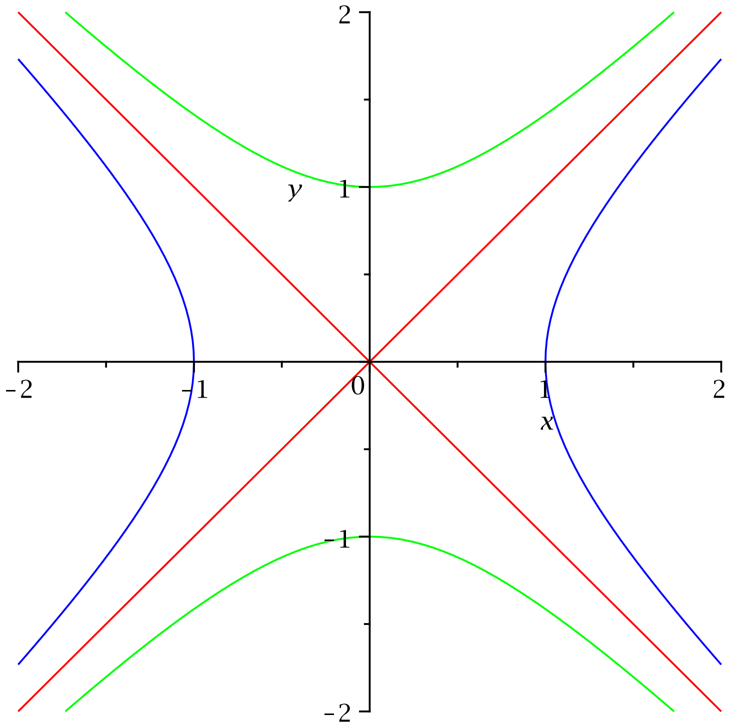


Figure: $\psi(\mathbf{r}, t)$ (in blue) and $j\psi(\mathbf{r}, t)$ (in green)

# Prediction of Antiparticles

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- The particles described by $\psi(\mathbf{r}, t)$ and $j\psi(\mathbf{r}, t)$, resp. have the same mass and the same energy. Their only difference is the sign of the charge. If one has charge $e$, the other has charge $-e$ and vice versa.
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- If we interpret the squared norm of the plane wave not just as the probability density but as the *unit charge probability density*, we arrive at the conclusion that the particles described by $\psi(\mathbf{r}, t)$ and $j\psi(\mathbf{r}, t)$, resp. are antiparticles of each other.
- The particles described by $\psi(\mathbf{r}, t)$ and $j\psi(\mathbf{r}, t)$, resp. have the same mass and the same energy. Their only difference is the sign of the charge. If one has charge $e$, the other has charge $-e$ and vice versa.
- Split-Complex QM predicts the existence of antiparticles at non-relativistic level. There is more. It also predicts the existence of parallel universes!

# Twin Universes

- The particles described by $\psi(\mathbf{r}, t)$ reside in $\mathbb{R}^{3+1}$ with signature $(+---)$ while their antiparticles described by $j\psi(\mathbf{r}, t)$ reside in $\mathbb{R}^{3+1}$ with signature $(-+--)$.
- I speculate that in the beginning the same amount of matter and antimatter were created by the Big Bang along with twin parallel universes, one mostly contains matter and the other mostly contains antimatter.
- This may explain the mysterious baryon asymmetry i.e. why antiparticles are so rare in the universe.

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# Split-Complex Structure and the Charge Conjugation Map

- Define a linear endomorphism $\mathscr{J} : \mathbb{C}' \longrightarrow \mathbb{C}'$ by

$$\mathscr{J}1 = j,\ \ \mathscr{J}j = 1$$

- $\mathscr{J}$ satisfies

$$\mathscr{J}^2 = \mathscr{I},\ \langle \mathscr{J}\zeta_1, \mathscr{J}\zeta_2\rangle = -\langle \zeta_1, \zeta_2\rangle$$

  Thus $\mathscr{J}$ is an anti-isometry. $\mathscr{J}$ is called the *associated split-complex structure* of $\mathbb{C}'$.

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# Path Integral Redux

- Split-Complex QM also remedies the formentioned issues with the path integral.
- The amplitude of a particle to propagate from a point $q_I$ to a point $q_F$ in time $T$ is obtained as

$$\langle q_F | e^{-\frac{j}{\hbar}\hat{H}T} | q_I \rangle = \int Dq(t) e^{\frac{j}{\hbar}\int_0^T dt L(\dot{q},q)}$$

- The Feynman meassure $Dq(t)$ is given by

$$\int Dq(t) := \lim_{N\to\infty} \left(\frac{2\pi m\hbar j}{\delta t}\right)^{\frac{N}{2}} \left(\prod_{k=1}^{N-1} \int dq_k\right)$$

where $\delta t = \frac{T}{N}$.

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# Path Integral Redux

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- Under the transformation $\mathbb{R}^{3+1} \longrightarrow \mathbb{R}^{3+1}$; $(t, x, y, z) \mapsto (-jt, x, y, z)$, the path integral turns into

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# Tachyon Equation

- Tachyons are hypothetical particles that travel faster than the speed of light and they expect to have imaginary rest mass.
- The energy-momentum relation for a tachyon is given by

$$E^2 = p^2c^2 - m_0^2c^4$$

- There is no equation for tachyons in conventional quantum physics. However, a Dirac type equation for tachyons can be obtained in Split-Complex QM. Further details about it will have to be a discussion for another time.

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# Questions?

Any Questions?