Spacelike surfaces of revolution with constant mean curvature in de Sitter 3-Space

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Abstract

In this paper, we construct spacelike surfaces of revolution with constant mean curvature $H = c$ and maximal spacelike surfaces of revolution in de Sitter 3-space $\mathbb{S}^3_1(c^2)$ of constant sectional curvature c^2. It is shown that surfaces of revolution with constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$ tend toward the maximal catenoid, the maximal spacelike surface of revolution in Minkowski 3-space $\mathbb{R}^{2+1}$ as $c \to 0$. Maximal spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ also tend toward the maximal catenoid in $\mathbb{R}^{2+1}$ as $c \to 0$.

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Introduction

Spacelike surfaces of constant mean curvature $H = c$ in de Sitter 3-space $\mathbb{S}^3_1(c^2)$ of constant sectional curvature c^2 share many geometric properties in common with maximal spacelike surfaces in Minkowski 3-space $\mathbb{R}^{2+1}$ ([1], [7]), although they live in two different spaces. It is not a coincidence. It turns out that there is an analogue of the *Lawson correspondence* ([6]) between spacelike surfaces of constant mean curvature H_s in $\mathbb{S}^3_1(c^2)$ and spacelike surfaces of constant mean curvature $H_m = \pm\sqrt{H_h^2 - c^2}$ in $\mathbb{R}^{2+1}$ ([13]). In particular, there is a Lawson type correspondence between spacelike surfaces of constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$ and maximal spacelike surfaces (i.e. conformal spacelike surfaces of constant mean curvature $H = 0$) in $\mathbb{R}^{2+1}$. These corresponding constant mean curvature surfaces satisfy the same Gauss-Codazzi equations. $\mathbb{S}^3_1(c^2)$ has a rotational symmetry, in fact SO(2) symmetry as its maximum rotational symmetry. So we may consider surfaces of revolution, in particular with constant mean curvature $H = c$. Spacelike surfaces of constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$ can be in general constructed by Bryant type representation formula ([1], [7]) using a holomorphic and a meromorphic functions analogously to the Weierstrass representation formula for maximal surfaces in $\mathbb{R}^{2+1}$ ([5], [11]), however it is not suitable to use to construct surfaces of revolution with constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$.

In section 1, we introduce the *flat chart model* of $\mathbb{S}^3_1(c^2)$. The flat chart model is convenient in many respects for our study of surfaces of revolution in $\mathbb{S}^3_1(c^2)$. In section 2, we calculate the mean curvature of a parametric spacelike surface in $\mathbb{S}^3_1(c^2)$. In section 4, we use this mean curvature formula to obtain the differential equation of the profile curve for a spacelike surface of revolution with constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$. The differential equation is nonlinear and it cannot be solved analytically. By solving this equation numerically, we construct a surface of revolution with constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$. In [1] and [7], it is shown that a maximal spacelike surface in $\mathbb{R}^{2+1}$ is the limit of spacelike surfaces of constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$ as $c \to 0$ using the deformation of Lie groups. In section 4, it is shown that spacelike surfaces of revolution with constant mean curvature $H = c$ tend toward the spacelike catenoid, the maximal spacelike surface of revolution in $\mathbb{R}^{2+1}$ as $c \to 0$ in a trivial manner from the differential equation. In section 5, we illustrate the limiting behavior with graphics.

Maximal spacelike surfaces in $\mathbb{S}^3_1(c^2)$ are not characterized by mean curvature unlike maximal spacelike surfaces in $\mathbb{R}^{2+1}$. In section 6, we construct maximal spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ using the calculus of variations. The maximal spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ also tend toward the spacelike catenoid in $\mathbb{R}^{2+1}$ as $c \to 0$.

1 The Flat Chart Model of de Sitter 3-Space $\mathbb{S}^3_1(c^2)$

Let $\mathbb{R}^{3+1}$ denote the Minkowski spacetime with rectangular coordinates x^0, x^1, x^2, x^3 and the Lorentzian metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{1}$$

de Sitter 3-space $\mathbb{S}^3_1(c^2)$ is the hyperquadric

$$\mathbb{S}^3_1(c^2) := \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^{3+1} : -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = \frac{1}{c^2}\} \tag{2}$$

which is a 3-dimensional hyperboloid of one sheet in spacetime. It is a timelike 3-manifold of constant sectional curvature c^2. Consider the open chart

$$U = \{(x^0, x^1, x^2, x^3) \in \mathbb{S}^3_1(c^2) : x^0 + x^1 > 0\}$$

and define (see [4])

$$\begin{aligned} t &= \frac{1}{c}\log c(x^0+x^1), \\ x &= \frac{x^2}{c(x^0+x^1)}, \\ y &= \frac{x^3}{c(x^0+x^1)}. \end{aligned} \tag{3}$$

Then

$$ds^2 = -(dt)^2 + e^{2ct}\{(dx)^2+(dy)^2\}.$$

$\mathbb{R}^3$ with coordinates t, x, y and the metric

$$g_c := -(dt)^2 + e^{2ct}\{(dx)^2+(dy)^2\} \tag{4}$$

is called the *flat chart model* of de Sitter 3-space. We will still denote it by $\mathbb{S}^3_1(c^2)$. As $c \to 0$, $\mathbb{S}^3_1(c^2)$ flattens out to Minkowski 3-space $\mathbb{R}^{2+1}$.

2 Parametric Spacelike Surfaces in $\mathbb{S}^3_1(c^2)$

Let M be a domain[1] and $\varphi : M \longrightarrow \mathbb{S}^3_1(c^2)$ an immersion. The metric (4) induces an inner product $\langle\ ,\ \rangle$ on each tangent space $T_p\mathbb{S}^3_1(c^2)$.

Definition 1. *An immersion $\varphi : M(u,v) \longrightarrow \mathbb{S}^3_1(c^2)$ is said to be* spacelike *if both the tangent vectors $\frac{\partial\varphi}{\partial u}$ and $\frac{\partial\varphi}{\partial v}$ are spacelike, i.e.*

$$\left\langle \frac{\partial\varphi}{\partial u}, \frac{\partial\varphi}{\partial u} \right\rangle > 0,\ \left\langle \frac{\partial\varphi}{\partial v}, \frac{\partial\varphi}{\partial v} \right\rangle > 0.$$

Using the induced inner product on each $T_p\mathbb{S}^3_1(c^2)$, we can speak of conformal surfaces in $\mathbb{S}^3_1(c^2)$.

Definition 2. *$\varphi : M \longrightarrow \mathbb{S}^3_1(c^2)$ is said to be* conformal *if*

$$\begin{aligned} \langle \varphi_u, \varphi_v \rangle &= 0, \\ |\varphi_u| = |\varphi_v| &= e^{\omega/2}, \end{aligned} \tag{5}$$

where (u,v) is a local coordinate system in M and $\omega : M \longrightarrow \mathbb{R}$ is a real-valued function in M.

The induced metric on the spacelike surface is given by

$$ds^2_\varphi = \langle d\varphi, d\varphi \rangle = e^\omega\{(du)^2+(dv)^2\}. \tag{6}$$

If N is a unit normal vector field of a spacelike immersion $\varphi : M \longrightarrow \mathbb{S}^3_1(c^2)$, then

$$\langle N, N \rangle = -1,\ \langle N, \varphi_u \rangle = \langle N, \varphi_v \rangle = 0.$$

[1] A 2-dimensional connected open set.

In order to calculate a unit normal vector field, we need an analogue of the cross product of vectors in Euclidean 3-space $\mathbb{R}^3$. We will call such an analogue still cross product. Although $\mathbb{S}^3_1(c^2)$ is not a vector space, cross product can be defined locally on each tangent space $T_p\mathbb{S}^3_1(c^2)$ which is a vector space. Let $\mathbf{v} = v_1\left(\frac{\partial}{\partial t}\right)_p + v_2\left(\frac{\partial}{\partial x}\right)_p + v_3\left(\frac{\partial}{\partial y}\right)_p$, $\mathbf{w} = w_1\left(\frac{\partial}{\partial t}\right)_p + w_2\left(\frac{\partial}{\partial x}\right)_p + w_3\left(\frac{\partial}{\partial y}\right)_p \in T_p\mathbb{S}^3_1(c^2)$, where $\left\{\left(\frac{\partial}{\partial t}\right)_p, \left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p\right\}$ denote the canonical basis for $T_p\mathbb{S}^3_1(c^2)$. Then the cross product $\mathbf{v}\times\mathbf{w}$ is defined to be

$$\begin{aligned}\mathbf{v}\times\mathbf{w} = (-v_2w_3 + v_3w_2)\left(\frac{\partial}{\partial t}\right)_p &+ e^{-2ct}(v_3w_1 - v_1w_3)\left(\frac{\partial}{\partial x}\right)_p \\ &+ e^{-2ct}(v_1w_2 - v_2w_1)\left(\frac{\partial}{\partial y}\right)_p\end{aligned} \tag{7}$$

where $p = (t, x, y) \in \mathbb{S}^3_1(c^2)$. We can also write (7) simply as a determinant

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} -\frac{\partial}{\partial t} & e^{-2ct}\frac{\partial}{\partial x} & e^{-2ct}\frac{\partial}{\partial y} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{8}$$

One may also define a triple scalar product $\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle$ as a determinant

$$\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle = \begin{vmatrix} -u_1 & e^{-2ct}u_2 & e^{-2ct}u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{9}$$

However, as one can clearly see the cross product and the inner product is not interchangeable i.e.

$$\langle \mathbf{u}, \mathbf{v}\times\mathbf{w}\rangle \neq \langle \mathbf{u}\times\mathbf{v}, \mathbf{w}\rangle$$

unlike Euclidean case.

Let

$$E := \langle \varphi_u, \varphi_u\rangle,\ F := \langle \varphi_u, \varphi_v\rangle,\ G := \langle \varphi_v, \varphi_v\rangle. \tag{10}$$

Proposition 1. *Let $\varphi : M \longrightarrow \mathbb{S}^3_1(c^2)$ be an immersion. Then on each tangent plane $T_p\varphi(M)$,*

$$|\varphi_u \times \varphi_v|^2 = e^{-4ct(u,v)}(F^2 - EG) \tag{11}$$

where $p = (t(u,v), x(u,v), y(u,v)) \in \mathbb{S}^3_1(c^2)$.

Proof. Straightforward by a direct calculation. □

Remark 1. If $c \to 0$, (11) becomes the familiar formula in Lorentzian case [10]

$$|\varphi_u \times \varphi_v|^2 = F^2 - EG.$$

Remark 2. The normal vector field $\varphi_u \times \varphi_v$ is timelike i.e. $F^2 - EG < 0$. So, the norm $|\varphi_u \times \varphi_v|$ is defined to be the *proper time*

$$|\varphi_u \times \varphi_v| := e^{-2ct(u,v)}\sqrt{EG - F^2}. \tag{12}$$

Accordingly, the unit normal vector field N of φ is given by

$$N = \frac{\varphi_u \times \varphi_v}{e^{-2ct(u,v)}\sqrt{EG - F^2}}. \tag{13}$$

In physics, the trajectory of a massive particle is a timelike curve in spacetime. The proper time is the actual time measured on a physical clock that is carried along the timelike curve.

3 The Mean curvature of a Parametric Surface in $\mathbb{S}^3_1(c^2)$

In Euclidean case, the mean curvature of a parametric surface $\varphi(u, v)$ may be calculated by the Gauss' beautiful formula [12]

$$H = \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG - F^2)} \tag{14}$$

where

$$\ell = \langle \varphi_{uu}, N \rangle,\ \mathfrak{m} = \langle \varphi_{uv}, N \rangle,\ \mathfrak{n} = \langle \varphi_{vv}, N \rangle$$

and N is the unit normal vector field of $\varphi(u, v)$. (14) is still valid for parametric surfaces in any 3-dimensional space including $\mathbb{S}^3_1(c^2)$. For the proof, see for instance Appendix A of [9].

Let $\varphi : M \longrightarrow \mathbb{S}^3_1(c^2)$ be a conformal spacelike surface satisfying (5) and N a unit normal vector field of φ. Let $S_p : T_p\varphi(M) \longrightarrow T_p\varphi(M)$ be the shape operator given by $S_p(\mathbf{v}) = -\nabla_{\mathbf{v}} N$ for $\mathbf{v} \in T_p\varphi(M)$. Let $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix associated with shape operator with respect to the orthogonal basis φ_u, φ_v of $T_p\varphi(M)$. Then

$$\begin{aligned} S(\varphi_u) &= a\varphi_u + b\varphi_v, \\ S(\varphi_v) &= c\varphi_u + d\varphi_v. \end{aligned}$$

So,

$$\begin{aligned} \langle S(\varphi_u), \varphi_u \rangle + \langle S(\varphi_v), \varphi_v \rangle &= e^{\omega}(a + b) \\ &= e^{\omega}\mathrm{Tr}S \\ &= 2e^{\omega}H. \end{aligned}$$

On the other hand, for a fixed v_0, $\varphi(u, v_0)$ is a curve on the surface and consider N to be restricted on this curve. Then $S(\varphi_u) = -N_u$. Differentiating $\langle \varphi_u, N\rangle = 0$ with respect to u, we obtain

$$\begin{aligned}\langle \varphi_{uu}, N\rangle &= -\langle \varphi_u, N_u\rangle \\ &= \langle \varphi_u, S(\varphi_u)\rangle.\end{aligned}$$

Similarly, we also obtain $\langle S(\varphi_v), \varphi_v\rangle = \langle \varphi_{vv}, N\rangle$. Finally the mean curvature H is given by

$$\begin{aligned}H &= \frac{1}{2}e^{-\omega}(\langle \varphi_{uu}, N\rangle + \langle \varphi_{vv}, N\rangle) \\ &= \frac{1}{2}e^{-\omega}\langle \triangle\varphi, N\rangle\end{aligned}$$

where $\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Proposition 2. *Let $\varphi : M \longrightarrow \mathbb{S}^3_1(c^2)$ be a conformal spacelike surface satisfying* (5)*. Then the mean curvature H of φ is computed to be*

$$H = \frac{1}{2}e^{-\omega}\langle \triangle\varphi, N\rangle. \tag{15}$$

One can easily see that the formulas (14) and (15) coincide for conformal surfaces.

4 Spacelike Surfaces of Revolution with Constant Mean Curvature in $\mathbb{S}^3_1(c^2)$

There is an interesting one-to-one correspondence, so called the *Lawson correspondence* between constant mean curvature surfaces in different Riemannian space forms [6]. The correspondence is more than just a bijection. Those corresponding constant mean curvature surfaces satisfy the same Gauss-Codazzi equations, so they share many geometric properties in common, even though they live in different spaces. For this reason they are often called *cousins*. There is an analogue of the Lawson correspondence between spacelike constant mean curvature surfaces in different semi-Riemannian space forms[13]. There is a Lawson type correspondence between spacelike surfaces of constant mean curvature H_s in $\mathbb{S}^3_1(c^2)$ and spacelike surfaces of constant mean curvature[2]

$$H_m = \pm\sqrt{H_s^2 - c^2} \tag{16}$$

In particular, surfaces of constant mean curvature $H = \pm c$ in $\mathbb{S}^3_1(c^2)$ are cousins of maximal spacelike surfaces[3] in Minkowski 3-space $\mathbb{R}^{2+1}$. The equation (16)

[2] The choice of $\pm$ signs depends on the orientation of the surface.

[3] Area maximizing spacelike surfaces or equivalently conformal spacelike surfaces with zero mean curvature.

tells that there are no constant mean curvature spacelike surfaces in $\mathbb{R}^{2+1}$ that are corresponded to $H_s = 0$ spacelike surfaces in $\mathbb{S}^3_1(c^2)$. It can be easily shown that there are no conformal spacelike surfaces of revolution with $H = 0$ in $\mathbb{S}^3_1(c^2)$. (See Proposition 3 below.) Note however that this does not mean there are no maximal spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ in case readers are only familiar with maximal spacelike surfaces in Minkowski 3-space. A parametric spacelike surface is called a *harmonic map* if it is a critical point of the area functional. A harmonic map is called a maximal surface[4] if it is conformal. From (15) , we see that $H = 0$ if and only if $\triangle\varphi = 0$. In Minkowski 3-space, φ is harmonic if and only if $\triangle\varphi = 0$, so a conformal parametric spacelike surface in $\mathbb{R}^{2+1}$ is maximal if and only if $H = 0$. (See [5], [11] for more details about maximal spacelike surfaces and constant mean curvature surfaces in $\mathbb{R}^{2+1}$.) However, this is no longer true in $\mathbb{S}^3_1(c^2)$ because the Laplace equation $\triangle\varphi = 0$ is not the harmonic map equation in $\mathbb{S}^3_1(c^2)$ as shown in [8]. Maximal spacelike surfaces in $\mathbb{S}^3_1(c^3)$ can be constructed in general using the Weierstrass representation formula obtained in [8]. In Section 6, we study how to construct maximal surface of revolution in $\mathbb{S}^3_1(c^2)$ using the calculus of variations.

In this section, we are interested in constructing a spacelike surface of revolution with constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$ which corresponds to a maximal spacelike surface in $\mathbb{R}^{2+1}$.

From the metric (4), one can see that $\mathbb{S}^3_1(c^2)$ has SO(2) symmetry i.e. SO(2) is a subgroup of the isometry group of $\mathbb{S}^3_1(c^2)$ and it is the maximum rotational symmetry. More specifically, the rotations about the t-axis (i.e. rotations on the xy-plane) are the only type of Euclidean rotations that can be considered in $\mathbb{S}^3_1(c^2)$.

Consider a profile curve $\alpha(u) = (g(u), h(u), 0)$ in the tx-plane. Denote by $\varphi(u, v)$ the rotation of $\alpha(u)$ about t-axis through an angle v. Then

$$\varphi(u, v) = (g(u), h(u)\cos v, h(u)\sin v). \tag{17}$$

If $\dot{g}(u) = \frac{dg(u)}{du}$ is never 0, (17) has a parametrization of the form

$$\varphi(w, v) = (w, f(w)\cos v, f(w)\sin v).$$

Thus, without loss of generality we may assume that $g(u) = u$ in (17). The quantities E, F, G are calculated to be

$$\begin{aligned} E &= e^{2cu}\{-e^{-2cu} + \dot{h}(u)^2\}, \\ F &= 0, \\ G &= e^{2cu}h(u)^2. \end{aligned}$$

If we require $\varphi(u, v)$ to be conformal, then

$$-e^{-2cu} + \dot{h}(u)^2 = h(u)^2. \tag{18}$$

[4]It is an area maximizing surface.

The quantities $\ell, \mathfrak{m}, \mathfrak{n}$ are calculated to be

$$\begin{aligned}
\ell &= -\frac{\ddot{h}(u)h(u)}{\sqrt{h(u)^2(-e^{-2cu}+\dot{h}(u)^2)}},\\
\mathfrak{m} &= 0,\\
\mathfrak{n} &= \frac{h(u)^2}{\sqrt{h(u)^2(-e^{-2cu}+\dot{h}(u)^2)}}.
\end{aligned}$$

So the mean curvature H is calculated by

$$\begin{aligned}
H &= \frac{G\ell + E\mathfrak{n} - 2F\mathfrak{m}}{2(EG-F^2)}\\
&= \frac{1}{2}\frac{-h(u)\ddot{h}(u) - e^{-2cu} + \dot{h}(u)^2}{e^{2cu}(-e^{-2cu}+\dot{h}(u)^2)\sqrt{h(u)^2(-e^{-2cu}+\dot{h}(u)^2)}}.
\end{aligned}$$

With the conformality condition (18), H becomes

$$H = \frac{-\ddot{h}(u)+h(u)}{2e^{2cu}h(u)^3}. \tag{19}$$

Differentiating (18) with respect to u, we obtain

$$\dot{h}(u)(-\ddot{h}(u)+h(u)) = ce^{-2cu}. \tag{20}$$

It follows from (19) and (20) that if $H=0$ then $c=0$ and hence we have:

Proposition 3. *There are no conformal spacelike surface of revolution with $H=0$ in $\mathbb{S}^3_1(c^2)$.*

Remark 3. In [9], the authors mentioned that the Lawson correspondence implies that there is no surface in hyperbolic 3-space $\mathbb{H}^3(-c^2)$ with $H=0$ unless $c=0$ in which case the space is Euclidean 3-space $\mathbb{E}^3$ (p.208). This is incorrect. The Lawson correspondence only implies that there are no constant mean curvature surfaces in $\mathbb{E}^3$ that are corresponded to $H=0$ surfaces in $\mathbb{H}^3(-c^2)$. However, similarly to Proposition 3, it can be shown that there are no conformal surfaces of revolution with $H=0$ in $\mathbb{H}^3(-c^2)$.

Remark 4. Although $\mathbb{H}^3(-c^2)$ and $\mathbb{S}^3_1(c^2)$ do not admit conformal surfaces of revolution with $H=0$, there may be surfaces with $H=0$ in $\mathbb{H}^3(-c^2)$ and in $\mathbb{S}^3_1(c^2)$. The plane $(0,u,v)$ is a conformal surface with $H=0$ in $\mathbb{H}^3(-c^2)$ and also in $\mathbb{S}^3_1(c^2)$. The helicoid $(v, \sinh v\cos u, \sinh v\sin u)$ is a surface with $H=0$ in $\mathbb{H}^3(-c^2)$. While it is a conformal surface in Euclidean 3-space $\mathbb{E}^3$ (i.e. when $c\to 0$), it is not a conformal surface in $\mathbb{H}^3(-c^2)$ because $E=e^{-2cv}\cosh^2 u$, $F=0$, and $G=1+e^{-2cv}\sinh^u$. The helicoid $(v,\cosh v\cos u,\cosh v\sin u)$ is a spacelike surface with $H=0$ in $\mathbb{S}^3_1(c^2)$. While it is a conformal surface in Minkowski 3-space $\mathbb{R}^{2+1}$ (i.e. when $c\to 0$), it is not a conformal surface in $\mathbb{S}^3_1(c^2)$ because $E=e^{2cv}(\cosh^2 u-1)$, $F=0$, and $G=-1+e^{2cv}\cosh^2 u$.

Remark 5. If one considers the catenoid

$$\varphi(u,v) = (u, \cosh u \cos v, \cosh u \sin v)$$

in $\mathbb{H}^3(-c^2)$, it is not conformal in $\mathbb{H}^3(-c^2)$ since $E = 1 + e^{-2cu}\sinh^2 u$, $F = 0$, and $G = e^{-2cu}\cosh^2 u$. Its mean curvature is neither 0 nor constant. It is given by

$$H = \frac{\sinh(cu)}{\cosh u(1 + e^{-2cu}\sinh^2 u)^{\frac{3}{2}}}.$$

$\varphi(u,v)$ satisfies the equation $\varphi_{uu} + \varphi_{vv} = 0$. Note that this does not lead to $H = 0$ since $\varphi(u,v)$ is not conformal.

Remark 6. If one considers the spacelike catenoid

$$\psi(u,v) = (u, \sinh u \cos v, \sinh u \sin v)$$

in $\mathbb{S}^3_1(c^2)$, it is not conformal in $\mathbb{S}^3_1(c^2)$ since $E = -1 + e^{2cu}\cosh^2 u$, $F = 0$, and $G = e^{2cu}\sinh^2 u$. Its mean curvature is neither 0 nor constant. It is given by

$$H = \frac{\sinh(cu)}{|\sinh u|(-1 + e^{2cu}\cosh^2 u)^{\frac{3}{2}}}.$$

$\psi(u,v)$ satisfies the equation $\psi_{uu} + \psi_{vv} = 0$. However, this does not lead to $H = 0$ since $\psi(u,v)$ is not conformal.

Let $H = c$. Then (19) can be written as

$$\ddot{h}(u) - h(u) + 2ce^{2cu}h(u)^3 = 0. \tag{21}$$

Hence, constructing a surface of revolution with $H = c$ comes down to solving the second-order nonlinear differential equation (21). Unfortunately, we cannot solve (21) analytically, so we solve it numerically with the aid of MAPLE software. (See appendix 7 for details of the computational procedure.) In the next section, we show the graphics of the surface of revolution with constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$ that we obtained using the numerical solution of the differential equation (21). The conformality condition (18) may be used to determine initial conditions. For all the numerical solutions of (21) in this paper, we used the same initial conditions $h(0) = 1.5$ and $\dot{h}(0) = \sqrt{h(0)^2 + 1} = 1.802775638$.

If $c \to 0$, then (21) becomes

$$\ddot{h}(u) - h(u) = 0 \tag{22}$$

which is an equation of overdamped simple harmonic oscillator. (22) has the general solution

$$h(u) = c_1 \cosh u + c_2 \sinh u.$$

This $h(u)$ gives rise to a maximal spacelike surface of revolution in $\mathbb{R}^{2+1}$ which is called a *spacelike catenoid*. Figure 1 shows a spacelike catenoid with $h(0) = 1.5$ and $\dot{h}(0) = 1.802775638$.

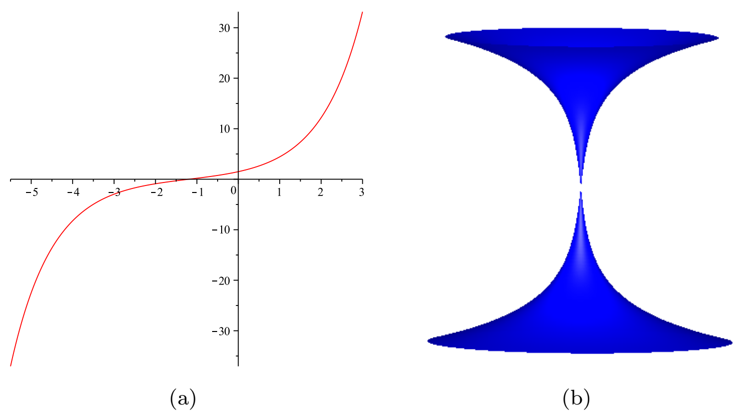


Fig. 1: (a) Profile Curve $h(u)$, $-5.5 \leq u \leq 3$, (b) Maximal Catenoid in $\mathbb{R}^{2+1}$

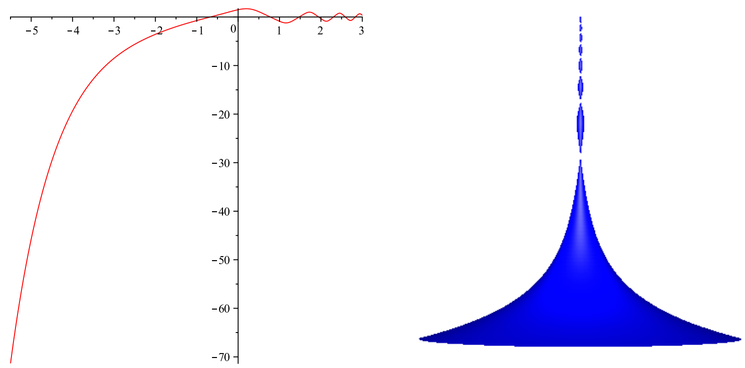
5 The Illustration of the Limit of Spacelike Surfaces of Revolution with $H = c$ in $\mathbb{S}^3_1(c^2)$ as $c \to 0$

In section 4, it is shown that the limit of conformal spacelike surfaces of revolution with constant mean curvature $H = c$ in $\mathbb{S}^3_1(c^2)$ is the spacelike catenoid, the maximal spacelike surface of revolution in $\mathbb{R}^{2+1}$. In this section, such limiting behavior of conformal spacelike surfaces of revolution with $H = c$ in $\mathbb{S}^3_1(c^2)$ is illustrated with graphics in Figure 2 ($H = 1$), Figure 3 ($H = \frac{1}{2}$), Figure 4 ($H = \frac{1}{4}$), Figure 5 ($H = \frac{1}{8}$), Figure 6 ($H = \frac{1}{16}$), Figure 7 ($H = \frac{1}{64}$), and Figure 8 ($H = \frac{1}{256}$). Figure 8 (b) already looks pretty close to the catenoid in Figure 1.

In order to visualize better the limiting behavior of surfaces of revolution with CMC $H = c$ in $\mathbb{S}^3_1(c^2)$ as $c \to 0$, the author has made some animations available in his website. An animation of profile curves for CMC $H = c$ spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ tending toward the profile curve of the catenoid in $\mathbb{R}^{2+1}$ as $c \to 0$ is available at `http://www.math.usm.edu/lee/sldsprofileanim.gif`. An animation of CMC $H = c$ spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ tending toward the spacelike catenoid in $\mathbb{R}^{2+1}$ as $c \to 0$ is available at `http://www.math.usm.edu/lee/sldscmcanim.gif`. The same animation of CMC $H = c$ spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ with the spacelike catenoid in $\mathbb{R}^{2+1}$ is available at `http://www.math.usm.edu/lee/sldscmcanim2.gif`.

6 Maximal Surface of Revolution in $\mathbb{S}^3_1(c^2)$

In section 4, we pointed out that maximal spacelike surfaces in $\mathbb{S}^3_1(c^2)$ are no longer characterized by mean curvature. In this section, we find the maximal spacelike surface of revolution in $\mathbb{S}^3_1(c^2)$ as a critical point of the area functional



(a) (b)

Fig. 2: CMC $H = 1$: (a) Profile Curve $h(u)$, $-5.5 \leq u \leq 3$, (b) Spacelike Surface of Revolution in $\mathbb{S}^3_1(1)$

using the calculus of variations.

Let us consider a surface of revolution which is obtained by revolving a curve $x(t)$ in the tx-plane about the t-axis. The curve is required to pass through the points (t_1, x_1) and (t_2, x_2) as seen in Figure 9. Our variational problem is to choose the curve $x(t)$ so that the area of the resulting surface of revolution is a maximum. The area element dA shown in Figure 9 is given by

$$dA = 2\pi x(t) ds = 2\pi x(t)\sqrt{-1 + e^{2ct}\dot{x}^2}dt, \tag{23}$$

where $\dot{x} = \frac{dx(t)}{dt}$. The area functional is then

$$J = \int_{t_1}^{t_2} 2\pi x(t)\sqrt{-1 + e^{2ct}\dot{x}^2}dt. \tag{24}$$

Let[5]

$$f(x, \dot{x}, t) = x\sqrt{-1 + e^{2ct}\dot{x}^2}.$$

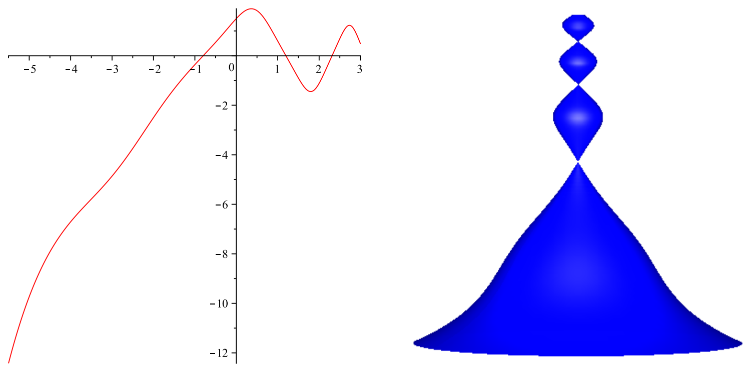
Finding a critical point of the area functional (24) is equivalent to solving the Euler-Lagrange equation (see [2] for example)

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} = 0 \tag{25}$$

which is equivalent to the second order nonlinear differential equation

$$-1 + e^{2ct}\dot{x}^2 + xce^{4ct}\dot{x}^3 - 2xce^{2ct}\dot{x} - xe^{2ct}\ddot{x} = 0. \tag{26}$$

[5]The constant 2π is neglected since it makes no contribution to the solution of our variational problem.



(a) (b)

Fig. 3: CMC $H = \frac{1}{2}$: (a) Profile Curve $h(u)$, $-5.5 \leq u \leq 3$, (b) Spacelike Surface of Revolution in $\mathbb{S}^3_1\left(\frac{1}{4}\right)$

Recall that a maximal spacelike surface is a conformal harmonic map. Applying the conformality condition (18), the equation (26) simplifies to

$$\ddot{x} - c(-1 + e^{2ct}x^2)\dot{x} - x = 0. \tag{27}$$

This nonlinear differential equation (27) cannot be solved analytically and again we need to solve it numerically. Figure 10 shows the profile curve $x(t)$ and the maximal spacelike surface of revolution in $\mathbb{S}^3_1(1)$. For the numerical solution, we also used the same initial conditions $x(0) = 1.5$ and $\dot{x}(0) = 1.802775638$ as before.

If $c \to 0$, then (27) becomes the equation of overdamped simple harmonic oscillator (22). Hence, as $c \to 0$ maximal spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ also tend toward the spacelike catenoid, the maximal spacelike surface of revolution in $\mathbb{R}^{2+1}$. An animation of this limiting behavior of maximal spacelike surfaces of revolution in $\mathbb{S}^3_1(c^2)$ is available at `http://www.math.usm.edu/lee/sldsmaximalanim.gif`.

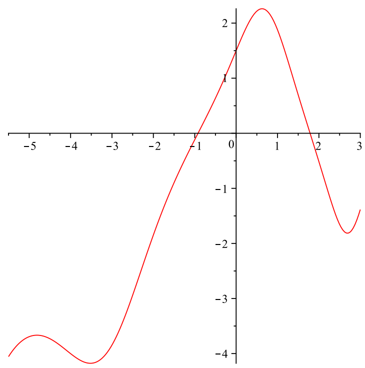
7 The Numerical Solution of (21) with MAPLE

The numerical solution of the differential equation (21) was obtained with the aid of MAPLE software version 15. For the readers who want to try by themselves, here are the MAPLE commands that the authors used to obtain the numerical solutions and the graphics. The commands need to be run in the following order.

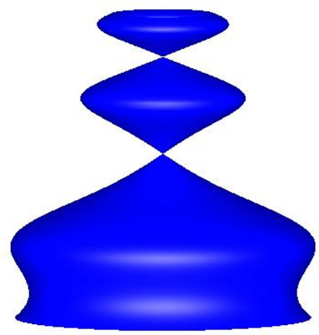
First we clear the memory.

```
restart:
```

In order to solve the equation numerically, we need a MAPLE package called DEtools.



(a)



(b)

Fig. 4: CMC $H = \frac{1}{4}$: (a) Profile Curve $h(u)$, $-5.5 \le u \le 3$, (b) Spacelike Surface of Revolution in $\mathbb{S}^3_1\left(\frac{1}{16}\right)$

`with(DEtools):`

Set the c value. In this example, we set $c = 1$.

`c:=1;`

Define the differential equation (21).

`eq:=diff(h(u),u,u)-h(u)+2*c*exp(2*c*u)*h(u)^3=0;`

Define the initial conditions for the equation (21).

`ic:=h(0)=1.5,D(h)(0)=1.802775638;`

Get the numerical solution.

`sol:=dsolve({eq,ic},numeric,output=listprocedure);`

Define the numerical solution as a function Y.

`Y:=subs(sol,h(u)):`

For testing, we evaluate $Y(.8)$.

`Y(.8);`

The output is

`-0.0927814313394189`

Now, we are ready to plot the profile curve $h(u)$.

`plot(Y,-5.5..3);`

The output is Figure 2 (a).

In order to plot surfaces, we need plot3d which is a part of the package called plots.

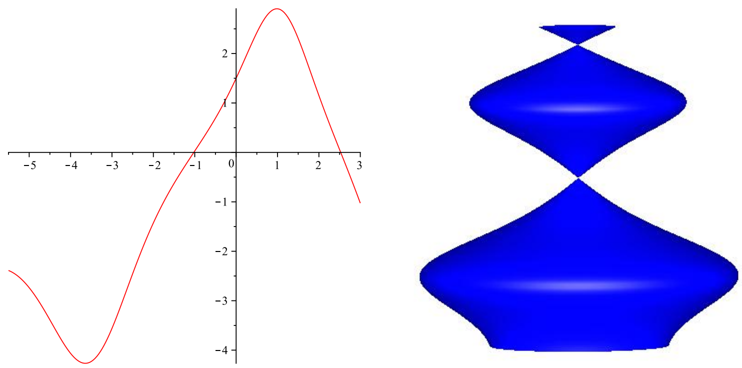
`with(plots);`

Define the surface of revolution X.

`X:=[u,Y(u)*cos(v),Y(u)*sin(v)];`

Finally, we plot the surface of revolution X.

```
plot3d(X,u=-5.5..3,v=0..2*Pi,grid=[85,85],style=patchnogrid,
shading=zhue,orientation=[62,64],color=blue,glossiness=1,
lightmodel=light1);
```



(a) (b)

Fig. 5: CMC $H = \frac{1}{8}$: (a) Profile Curve $h(u)$, $-5.5 \le u \le 3$, (b) Spacelike Surface of Revolution in $\mathbb{S}^3_1\left(\frac{1}{64}\right)$

The output is Figure 2 (b).

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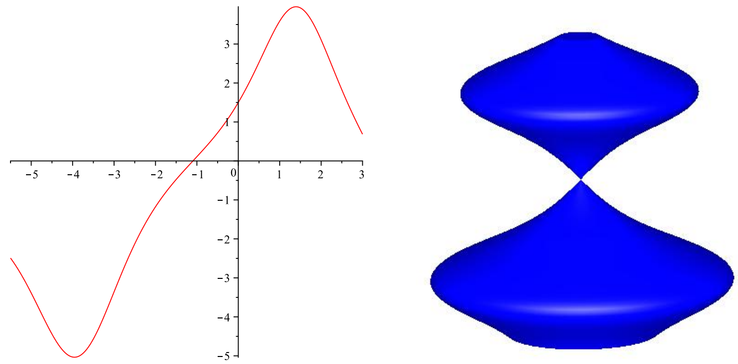

(a) (b)

Fig. 6: CMC $H = \frac{1}{16}$: (a) Profile Curve $h(u)$, $-5.5 \leq u \leq 3$, (b) Spacelike Surface of Revolution in $\mathbb{S}^3_1\left(\frac{1}{256}\right)$

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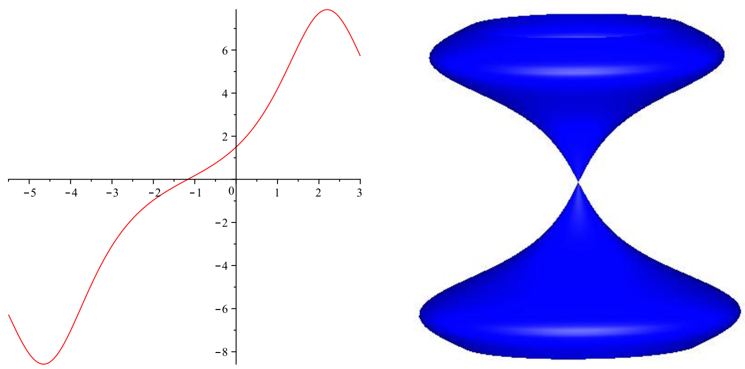
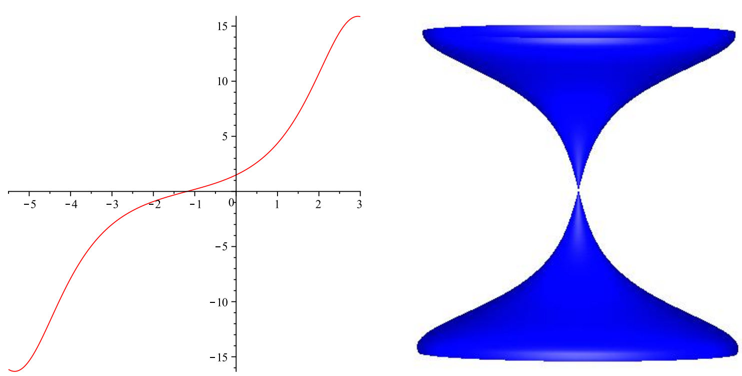


(a) (b)

Fig. 7: CMC $H = \frac{1}{64}$: (a) Profile Curve $h(u)$, $-5.5 \leq u \leq 3$, (b) Spacelike Surface of Revolution in $\mathbb{S}^3_1\left(\frac{1}{4096}\right)$



(a) (b)

Fig. 8: CMC $H = \frac{1}{256}$: (a) Profile Curve $h(u)$, $-5.5 \leq u \leq 3$, (b) Spacelike Surface of Revolution in $\mathbb{S}^3_1\left(\frac{1}{65535}\right)$

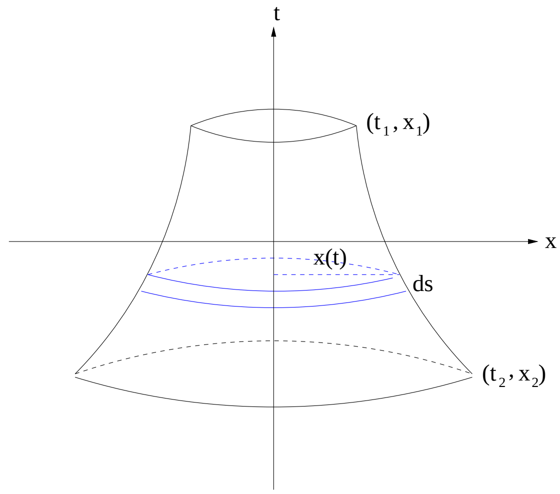


Fig. 9: Surface of Revolution in $\mathbb{S}^3_1(c^2)$

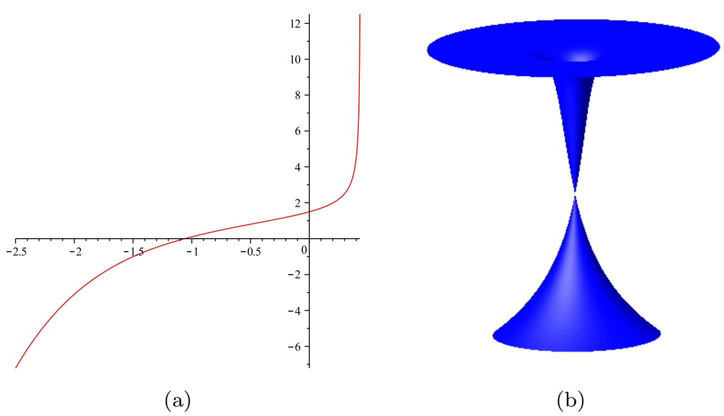


Fig. 10: (a) Profile Curve $x(t)$, $-2.5 \le t \le 0.43$, (b) Maximal Spacelike Surface of Revolution in $\mathbb{S}^3_1(1)$