

MAXIMAL SPACELIKE SURFACES IN A CERTAIN HOMOGENEOUS LORENTZIAN 3-MANIFOLD

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ABSTRACT. The 2-parameter family of certain homogeneous Lorentzian 3-manifolds, which includes Minkowski 3-space and anti-de Sitter 3-space, is considered. Each homogeneous Lorentzian 3-manifold in the 2-parameter family has a solvable Lie group structure with left invariant metric. A generalized integral representation formula for maximal spacelike surfaces in the homogeneous Lorentzian 3-manifolds is obtained. The normal Gauß map of maximal spacelike surfaces and its harmonicity are discussed.

INTRODUCTION

In [3]-[4], J. Inoguchi studied Weierstraß-Enneper formula for minimal surfaces in the 2-parameter family of Riemannian homogeneous spaces $(\mathbb{R}^3, g[\mu_1, \mu_2])$ with

$$g[\mu_1, \mu_2] = e^{-\mu_1 t} dx^2 + e^{-\mu_2 t} dy^2 + dt^2.$$

Here, μ_1, μ_2 are real constants. Every homogeneous Riemannian manifold in this family can be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Riemannian manifolds includes Euclidean 3-space and hyperbolic 3-space. Euclidean 3-space and hyperbolic 3-space are in fact the only homogeneous Riemannian manifolds in this family that have constant sectional curvature. The Weierstraß-Enneper formula obtained by Inoguchi is a generalized one that includes representation formulas for minimal surfaces in Euclidean 3-space, the well-known classical formula, and for minimal surfaces in hyperbolic 3-space obtained by M. Kokubu in [7] and independently by C. C. Góes and P. A. Q. Simões in [2]. The

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generalized Weierstraß-Enneper formula also contains an integral representation formula, obtained by Mercuri, Montaldo and Piu [10], for minimal surfaces in the Riemannian direct product $\mathbb{H}^2 \times \mathbb{E}^1$ of hyperbolic 2-space and the real line \mathbb{E}^1 . Minimal surfaces in $\mathbb{H}^2 \times \mathbb{E}^1$ were also studied by B. Nelli and H. Rosenberg in [11] and [12]. On the other hand, in [8], the author considered the 2-parameter family of homogeneous Lorentzian 3-manifolds $(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1, \mu_2)})$ with Lorentzian metric

$$g_{(\mu_1, \mu_2)} = -(dx^0)^2 + e^{-2\mu_1 x^0} (dx^1)^2 + e^{-2\mu_2 x^0} (dx^2)^2.$$

Every homogeneous Lorentzian 3-manifold in this family can be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Lorentzian 3-manifolds includes Minkowski 3-space \mathbb{E}_1^3 , de Sitter 3-space $\mathbb{S}_1^3(c^2)$ of constant sectional curvature c^2 , and $\mathbb{S}_1^2(c^2) \times \mathbb{E}^1$, the direct product of de Sitter 2-space $\mathbb{S}_1^2(c^2)$ of constant curvature c^2 and the real line \mathbb{E}^1 . (In the family, only Minkowski 3-space and de Sitter 3-space have constant sectional curvature.) These three spaces may be considered as Lorentzian counterparts of Euclidean 3-space \mathbb{E}^3 , hyperbolic 3-space $\mathbb{H}^3(-c^2)$, and the direct product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$, respectively, of Thurston's eight model geometries [13]. In [8], the author obtained a generalized integral representation formula that includes Weierstraß representation formula for maximal spacelike surfaces in Minkowski 3-space studied independently by O. Kobayashi [6] and by L. McNertney [9], and Weierstraß representation formula for maximal spacelike surfaces in de Sitter 3-space.

In this paper, we consider the 2-parameter family of homogeneous Lorentzian 3-manifolds $(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1, \mu_2)})$ with Lorentzian metric

$$g_{(\mu_1, \mu_2)} = -e^{-2\mu_1 x^2} (dx^0)^2 + e^{-2\mu_2 x^2} (dx^1)^2 + (dx^2)^2.$$

Every homogeneous Lorentzian manifold in this family can also be represented as a solvable matrix Lie group with left invariant metric. This family of homogeneous Lorentzian 3-manifolds includes Minkowski 3-space \mathbb{E}_1^3 , anti-de Sitter 3-space $\mathbb{H}_1^3(-c^2)$ of constant sectional curvature $-c^2$, $\mathbb{H}^2(-c^2) \times \mathbb{E}_1^1$, the direct product of hyperbolic plane $\mathbb{H}^2(-c^2)$ of constant curvature $-c^2$ and the timeline \mathbb{E}_1^1 , and $\mathbb{H}_1^2(-c^2) \times \mathbb{E}^1$, the direct product of anti-de Sitter 2-space $\mathbb{H}_1^2(-c^2)$ of constant curvature $-c^2$ and the real line \mathbb{E}^1 . (In the family, only Minkowski 3-space and anti-de Sitter 3-space have constant sectional curvature.) These four spaces may be considered as Lorentzian counterparts of Euclidean 3-space \mathbb{E}^3 , 3-sphere \mathbb{S}^3 , the direct product $\mathbb{H}^2(-c^2) \times \mathbb{E}^1$, and $\mathbb{S}^2 \times \mathbb{E}^1$,

the direct product of 2-sphere \mathbb{S}^2 and the real line \mathbb{E}^1 , respectively, of Thurston's eight model geometries [13]. We obtain a generalized integral representation formula that includes, in particular, representation formulas for maximal spacelike surfaces in Minkowski 3-space ([6], [9]) and in anti-de Sitter 3-space. The normal Gauß map of maximal spacelike surfaces in $G(\mu_1, \mu_2)$ is discussed. It is shown that Minkowski 3-space $G(0, 0)$, anti-de Sitter 3-space $G(c, c)$, and $G(c, -c)$ are the only homogeneous Lorentzian 3-manifolds among the 2-parameter family members $G(\mu_1, \mu_2)$ in which the (projected) normal Gauß map of maximal spacelike surfaces is harmonic. The harmonic map equations for those cases are also obtained.

1. SOLVABLE LIE GROUP

In this section, we study the following two-parameter family of homogeneous Lorentzian 3-manifolds;

$$(1) \quad \{(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1, \mu_2)}) \mid (\mu_1, \mu_2) \in \mathbb{R}^2\},$$

where the metrics $g_{(\mu_1, \mu_2)}$ are defined by

$$(2) \quad g_{(\mu_1, \mu_2)} := -e^{-2\mu_1 x^2} (dx^0)^2 + e^{-2\mu_2 x^2} (dx^1)^2 + (dx^2)^2.$$

Proposition 1. *Each homogeneous space $(\mathbb{R}^3, g_{(\mu_1, \mu_2)})$ is isometric to the following solvable matrix Lie group:*

$$G(\mu_1, \mu_2) = \left\{ \left(\begin{array}{cccc} e^{\mu_1 x^2} & 0 & 0 & x^0 \\ 0 & e^{\mu_2 x^2} & 0 & x^1 \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{array} \right) \mid x^0, x^1, x^2 \in \mathbb{R} \right\}$$

with left invariant metric. The group operation on $G(\mu_1, \mu_2)$ is the ordinary matrix multiplication and the corresponding group operation on $(\mathbb{R}^3, g_{(\mu_1, \mu_2)})$ is given by

$$(x^0, x^1, x^2) \cdot (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2) = (x^0 + e^{\mu_1 x^2} \tilde{x}^0, x^1 + e^{\mu_2 x^2} \tilde{x}^1, x^2 + \tilde{x}^2).$$

Proof. For $\tilde{a} = (a^0, a^1, a^2) \in G(\mu_1, \mu_2)$, denote by $L_{\tilde{a}}$ the left translation by \tilde{a} . Then

$$\begin{aligned} L_{\tilde{a}}(x^0, x^1, x^2) &= (a^0, a^1, a^2) \cdot (x^0, x^1, x^2) \\ &= (a^0 + e^{\mu_1 a^2} x^0, a^1 + e^{\mu_2 a^2} x^1, a^2 + x^2) \end{aligned}$$

and

$$\begin{aligned} L_{\tilde{a}}^*g(\mu_1, \mu_2) &= -e^{-2\mu_1(a^2+x^2)}\{d(a^0 + e^{\mu_1 a^2} x^0)\}^2 + \\ &\quad e^{-2\mu_2(a^2+x^2)}\{d(a^1 + e^{\mu_2 a^2} x^1)\}^2 + \{d(a^2 + x^2)\}^2 \\ &= -e^{-2\mu_1 x^2}(dx^0)^2 + e^{-2\mu_2 x^2}(dx^1)^2 + (dx^2)^2. \end{aligned}$$

This completes the proof. \square

The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is given explicitly by

$$(3) \quad \mathfrak{g}(\mu_1, \mu_2) = \left\{ \left(\begin{array}{cccc} \mu_1 y^2 & 0 & 0 & y^0 \\ 0 & \mu_2 y^2 & 0 & y^1 \\ 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & 0 \end{array} \right) \mid y^0, y^1, y^2 \in \mathbb{R} \right\}.$$

Then we can take the following orthonormal basis $\{E_0, E_1, E_2\}$ of $\mathfrak{g}(\mu_1, \mu_2)$:

$$(4) \quad E_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the commutation relation of $\mathfrak{g}(\mu_1, \mu_2)$ is given by

$$\begin{aligned} [E_0, E_1] &= 0, [E_1, E_2] = -\mu_2 E_1, \\ [E_2, E_0] &= \mu_1 E_0. \end{aligned}$$

$[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$, so $\mathfrak{g}(\mu_1, \mu_2)$ is a solvable Lie algebra i.e. $G(\mu_1, \mu_2)$ is a solvable Lie group. For $X \in \mathfrak{g}(\mu_1, \mu_2)$, denote by $\text{ad}(X)^*$ the *adjoint* operator of $\text{ad}(X)$. Then it satisfies the equation

$$\langle [X, Y], Z \rangle = \langle Y, \text{ad}(X)^*(Z) \rangle$$

for any $Y, Z \in \mathfrak{g}(\mu_1, \mu_2)$. Let U be the symmetric bilinear operator on $\mathfrak{g}(\mu_1, \mu_2)$ defined by

$$U(X, Y) := \frac{1}{2} \{ \text{ad}(X)^*(Y) + \text{ad}(Y)^*(X) \}.$$

Lemma 2. *Let $\{E_0, E_1, E_2\}$ be the orthonormal basis for $\mathfrak{g}(\mu_1, \mu_2)$ defined in (4). Then*

$$\begin{aligned} U(E_0, E_0) &= \mu_1 E_2, \quad U(E_1, E_1) = -\mu_2 E_2, \quad U(E_2, E_2) = 0, \\ U(E_0, E_1) &= 0, \quad U(E_1, E_2) = \frac{\mu_2}{2} E_1, \quad U(E_2, E_0) = \frac{\mu_1}{2} E_0. \end{aligned}$$

Lemma 3 (M. Kokubu [7], K. Uhlenbeck [14]). *Let \mathfrak{D} be a simply connected domain. A smooth map $\varphi : \mathfrak{D} \rightarrow G(\mu_1, \mu_2)$ is harmonic if and only if*

$$(5) \quad (\varphi^{-1}\varphi_u)_u + (\varphi^{-1}\varphi_v)_v - \text{ad}(\varphi^{-1}\varphi_u)^*(\varphi^{-1}\varphi_u) - \text{ad}(\varphi^{-1}\varphi_v)^*(\varphi^{-1}\varphi_v) = 0$$

holds.

Let $z = u + iv$. Then in terms of complex coordinates z, \bar{z} , the harmonic map equation (5) can be written as

$$(6) \quad \frac{\partial}{\partial \bar{z}} \left(\varphi^{-1} \frac{\partial \varphi}{\partial z} \right) + \frac{\partial}{\partial z} \left(\varphi^{-1} \frac{\partial \varphi}{\partial \bar{z}} \right) - 2U \left(\varphi^{-1} \frac{\partial \varphi}{\partial z}, \varphi^{-1} \frac{\partial \varphi}{\partial \bar{z}} \right) = 0.$$

Let $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$. Then the equation (6) is equivalent to

$$(7) \quad A_{\bar{z}} + \bar{A}_z = 2U(A, \bar{A}).$$

The Maurer-Cartan equation is given by

$$(8) \quad A_{\bar{z}} - \bar{A}_z = [A, \bar{A}].$$

The equations (7) and (8) can be combined to a single equation

$$(9) \quad A_{\bar{z}} = U(A, \bar{A}) + \frac{1}{2}[A, \bar{A}].$$

The equation (9) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = Adz + \bar{A}d\bar{z}$ and the condition for φ to be a harmonic map.

Left-translating the basis $\{E_0, E_1, E_2\}$, we obtain the following orthonormal frame field:

$$e_0 = e^{\mu_1 x^2} \frac{\partial}{\partial x^0}, \quad e_1 = e^{\mu_2 x^2} \frac{\partial}{\partial x^1}, \quad e_2 = \frac{\partial}{\partial x^2}.$$

The Levi-Civita connection ∇ of $G(\mu_1, \mu_2)$ is computed to be

$$\begin{aligned} \nabla_{e_0} e_0 &= -\mu_1 e_2, & \nabla_{e_0} e_1 &= 0, & \nabla_{e_0} e_2 &= -\mu_1 e_0, \\ \nabla_{e_1} e_0 &= 0, & \nabla_{e_1} e_1 &= \mu_2 e_2, & \nabla_{e_1} e_2 &= -\mu_2 e_1, \\ \nabla_{e_2} e_0 &= -\mu_1 e_0, & \nabla_{e_2} e_1 &= -\mu_2 e_1, & \nabla_{e_2} e_2 &= 0. \end{aligned}$$

Let $K(e_i, e_j)$ denote the sectional curvature of $G(\mu_1, \mu_2)$ with respect to the tangent plane spanned by e_i and e_j for $i, j = 0, 1, 2$. Then

$$(10) \quad \begin{aligned} K(e_0, e_1) &= g^{00} R_{010}^1 = -\mu_1 \mu_2, \\ K(e_1, e_2) &= g^{11} R_{121}^2 = -\mu_2^2, \\ K(e_0, e_2) &= g^{00} R_{020}^3 = -\mu_1^2, \end{aligned}$$

where $g_{ij} = g_{(\mu_1, \mu_2)}(e_i, e_j)$ denotes the metric tensor of $G(\mu_1, \mu_2)$. Hence, we see that $G(\mu_1, \mu_2)$ has a constant sectional curvature if and only if $\mu_1^2 = \mu_2^2 = \mu_1\mu_2$. If $c := \mu_1 = \mu_2$, then $G(\mu_1, \mu_2)$ is locally isometric to $\mathbb{H}_1^3(-c^2)$, the anti-de Sitter 3-space of constant sectional curvature $-c^2$. (See Example 2 and Remark 1.) If $G(\mu_1, \mu_2)$ has a constant sectional curvature and $\mu_1 = -\mu_2$, then $\mu_1 = \mu_2 = 0$, so $G(\mu_1, \mu_2) = G(0, 0) \cong \mathbb{E}_1^3$ (Example 1).

Example 1. (Minkowski 3-space) The Lie group $G(0, 0)$ is isomorphic and isometric to the Minkowski 3-space

$$\mathbb{E}_1^3 = (\mathbb{R}^3(x^0, x^1, x^2), +)$$

with the metric $-(dx^0)^2 + (dx^1)^2 + (dx^2)^2$.

Example 2. (Anti-de Sitter 3-space) Take $\mu_1 = \mu_2 = c \neq 0$. Then $G(c, c)$ is the flat chart model of the anti-de Sitter 3-space:

$$\mathbb{H}_1^3(-c^2)_+ = (\mathbb{R}^3(x^0, x^1, x^2), e^{-2cx^2}\{-(dx^0)^2 + (dx^1)^2\} + (dx^2)^2).$$

Remark 1. Let \mathbb{E}_2^4 be the pseudo-Euclidean 4-space with the metric $\langle \cdot, \cdot \rangle$:

$$\langle \cdot, \cdot \rangle = -(du^0)^2 - (du^1)^2 + (du^2)^2 + (du^3)^2.$$

in terms of rectangular coordinate system (u^0, u^1, u^2, u^3) . The *anti-de Sitter 3-space* $\mathbb{H}_1^3(-c^2)$ of constant sectional curvature $-c^2$ is realized as the hyperquadric in \mathbb{E}_2^4 :

$$\mathbb{H}_1^3(-c^2) = \left\{ (u^0, u^1, u^2, u^3) \in \mathbb{E}_2^4 : -(u^0)^2 - (u^1)^2 + (u^2)^2 + (u^3)^2 = -\frac{1}{c^2} \right\}.$$

The anti-de Sitter 3-space $\mathbb{H}_1^3(-c^2)$ is divided into the following three regions:

$$\mathbb{H}_1^3(-c^2)_+ = \{(u^0, u^1, u^2, u^3) \in \mathbb{H}_1^3(-c^2) : c(u^1 + u^2) > 0\};$$

$$\mathbb{H}_1^3(-c^2)_0 = \{(u^0, u^1, u^2, u^3) \in \mathbb{H}_1^3(-c^2) : u^1 + u^2 = 0\};$$

$$\mathbb{H}_1^3(-c^2)_- = \{(u^0, u^1, u^2, u^3) \in \mathbb{H}_1^3(-c^2) : c(u^1 + u^2) < 0\}.$$

$\mathbb{H}_1^3(-c^2)$ is the disjoint union $\mathbb{H}_1^3(-c^2)_+ \dot{\cup} \mathbb{H}_1^3(-c^2)_0 \dot{\cup} \mathbb{H}_1^3(-c^2)_-$ and $\mathbb{H}_1^3(-c^2)_\pm$ are diffeomorphic to $(\mathbb{R}^3, g_{(c,c)})$. Let us introduce a local

coordinate system (x^0, x^1, x^2) on $\mathbb{H}_1^3(-c^2)_+$ by

$$\begin{aligned} x^0 &= \frac{u^0}{c(u^1 + u^2)}, \\ x^1 &= \frac{u^3}{c(u^1 + u^2)}, \\ x^2 &= -\frac{1}{c} \ln[c(u^1 + u^2)]. \end{aligned}$$

The induced metric of $\mathbb{H}_1^3(-c^2)_+$ is expressed as:

$$g_c := e^{-2cx^2} \{-(dx^0)^2 + (dx^1)^2\} + (dx^2)^2.$$

The chart $(\mathbb{H}_1^3(-c^2)_+, g_c)$ is called the *flat chart* of $\mathbb{H}_1^3(-c^2)$. The flat chart is identified with the Lorentzian manifold $(\mathbb{R}^3, g_{(c,c)})$ of constant sectional curvature $-c^2$. This expression shows that the flat chart is a warped product $\mathbb{E}^1 \times_f \mathbb{E}_1^2$ with warping function $f(x^2) = e^{-cx^2}$. Introducing $y^0 = cx^0$, $y^1 = cx^1$, and $y^2 = e^{cx^2}$, we also obtain half-space model of anti-de Sitter 3-space $\mathbb{H}_1^3(-c^2)$ with an analogue of Poincaré metric

$$g_c := \frac{-(dy^0)^2 + (dy^1)^2 + (dy^2)^2}{c^2(y^2)^2}.$$

Example 3 (Direct Product $\mathbb{H}^2(-c^2) \times \mathbb{E}_1^1$). Take $(\mu_1, \mu_2) = (0, c)$ with $c \neq 0$. Then the resulting homogeneous spacetime is \mathbb{R}^3 with the Lorentzian metric

$$-(dx^0)^2 + e^{-2cx^2} (dx^1)^2 + (dx^2)^2.$$

$G(0, c)$ is identified with $\mathbb{H}^2(-c^2) \times \mathbb{E}_1^1$, the direct product of hyperbolic plane $\mathbb{H}^2(-c^2)$ of constant curvature $-c^2$ and the timeline \mathbb{E}_1^1 .

Example 4 (Direct Product $\mathbb{H}_1^2(-c^2) \times \mathbb{E}^1$). Take $(\mu_1, \mu_2) = (c, 0)$ with $c \neq 0$. Then the resulting homogeneous spacetime is \mathbb{R}^3 with the Lorentzian metric

$$-e^{-2cx^2} (dx^0)^2 + (dx^2)^2 + (dx^1)^2.$$

$G(c, 0)$ is identified with $\mathbb{H}_1^2(-c^2) \times \mathbb{E}^1$, the direct product of anti-de Sitter 2-space $\mathbb{H}_1^2(-c^2)$ of constant curvature $-c^2$ and the real line \mathbb{E}^1 .

Example 5 (Homogeneous Spacetime $G(c, -c)$). Let $\mu_1 = c$ and $\mu_2 = -c$ with $c \neq 0$. Then the resulting homogeneous spacetime $G(c, -c)$ is \mathbb{R}^3 with the Lorentzian metric

$$-e^{-2cx^2} (dx^0)^2 + e^{2cx^2} (dx^1)^2 + (dx^2)^2.$$

2. INTEGRAL REPRESENTATION FORMULA

In this section, we obtain a general integral representation formula for maximal spacelike surfaces in $G(\mu_1, \mu_2)$ analogously to [3] and [8].

Let $\mathfrak{D}(z, \bar{z})$ be a simply connected domain and $\varphi : \mathfrak{D} \rightarrow G(\mu_1, \mu_2)$ a smooth map. If we write $\varphi(z) = (x^0(z), x^1(z), x^2(z))$ then by direct calculation

$$A = x_z^0 e^{-\mu_1 x^2} E_0 + x_z^1 e^{-\mu_2 x^2} E_1 + x_z^2 E_2.$$

It follows from the harmonic map equation (7) that

Lemma 4. *φ is harmonic if and only if the following equations hold:*

$$\begin{aligned} x_{z\bar{z}}^0 - \mu_1(x_z^0 x_z^2 + x_z^0 x_z^2) &= 0, \\ x_{z\bar{z}}^1 - \mu_2(x_z^1 x_z^2 + x_z^1 x_z^2) &= 0, \\ x_{z\bar{z}}^2 - \mu_1 x_z^0 x_z^0 e^{-2\mu_1 x^2} + \mu_2 x_z^1 x_z^1 e^{-2\mu_2 x^2} &= 0. \end{aligned}$$

The exterior derivative d is decomposed as

$$d = \partial + \bar{\partial}, \quad \partial = \frac{\partial}{\partial z} dz, \quad \bar{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z},$$

with respect to the conformal structure of \mathfrak{D} . Let $\omega^0 = e^{-\mu_1 x^2} x_z^0 dz$, $\omega^1 = e^{-\mu_2 x^2} x_z^1 dz$, $\omega^2 = x_z^2 dz$. Then by Lemma 4, the triplet $\{\omega^0, \omega^1, \omega^2\}$ of (1,0)-forms satisfies the following differential system:

$$(11) \quad \bar{\partial} \omega^i = \mu_{i+1} \bar{\omega}^i \wedge \omega^2, \quad i = 0, 1,$$

$$(12) \quad \bar{\partial} \omega^2 = \mu_1 \bar{\omega}^0 \wedge \omega^0 - \mu_2 \bar{\omega}^1 \wedge \omega^1.$$

Proposition 5. *Let $\{\omega^0, \omega^1, \omega^2\}$ be a solution to (11)-(12) on a simply connected domain \mathfrak{D} . Then*

$$\varphi(z, \bar{z}) = 2\text{Re} \int_{z_0}^z \left(e^{\mu_1 x^2(z, \bar{z})} \cdot \omega^0, e^{\mu_2 x^2(z, \bar{z})} \cdot \omega^1, \omega^2 \right)$$

is a harmonic map into $G(\mu_1, \mu_2)$.

Conversely, any harmonic map of \mathfrak{D} into $G(\mu_1, \mu_2)$ can be represented in this form.

Corollary 6. *Let $\{\omega^0, \omega^1, \omega^2\}$ be a solution to (11)-(12) along with*

$$(13) \quad -\omega^0 \otimes \omega^0 + \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 = 0$$

on a simply connected domain \mathfrak{D} . Then

$$\varphi(z, \bar{z}) = 2\text{Re} \int_{z_0}^z \left(e^{\mu_1 x^2(z, \bar{z})} \cdot \omega^0, e^{\mu_2 x^2(z, \bar{z})} \cdot \omega^1, \omega^2 \right)$$

is a weakly conformal harmonic map into $G(\mu_1, \mu_2)$. Moreover $\varphi(z, \bar{z})$ is a maximal spacelike surface¹ if

$$-\omega^0 \otimes \bar{\omega}^0 + \omega^1 \otimes \bar{\omega}^1 + \omega^2 \otimes \bar{\omega}^2 \neq 0.$$

3. THE NORMAL GAUSS MAP

Let $\varphi : \mathfrak{D} \rightarrow G(\mu_1, \mu_2)$ be a conformal surface. Take the future-pointing unit normal N along φ . Then, by the left translation we obtain the following smooth map:

$$\varphi^{-1} \cdot N : \mathfrak{D} \rightarrow \mathbb{H}^2(-1),$$

where

$$\begin{aligned} \mathbb{H}^2(-1) &= \{u^0 E_0 + u^1 E_1 + u^2 E_2 : -(u^0)^2 + (u^1)^2 + (u^2)^2 = -1, u^0 > 0\} \\ &\subset \mathfrak{g}(\mu_1, \mu_2) \end{aligned}$$

is the unit hyperbolic 2-space. The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is identified with Minkowski 3-space $\mathbb{E}_1^3(u^0, u^1, u^2)$ via the orthonormal basis $\{E_0, E_1, E_2\}$. The smooth map $\varphi^{-1} \cdot N$ is called the *normal Gauß map* of φ .

Let $\varphi : \mathfrak{D} \rightarrow G(\mu_1, \mu_2)$ be a maximal spacelike immersion of a simply connected Riemann surface \mathfrak{D} determined by the data $(\omega^0, \omega^1, \omega^2)$. Write the data as $\omega^i = \psi^i dz$, $i = 0, 1, 2$. Then the induced metric I of φ is

$$\begin{aligned} (14) \quad I &= 2(-\omega^0 \otimes \bar{\omega}^0 + \omega^1 \otimes \bar{\omega}^1 + \omega^2 \otimes \bar{\omega}^2) \\ &= 2(-|\psi^0|^2 + |\psi^1|^2 + |\psi^2|^2) dz d\bar{z}. \end{aligned}$$

From the conformality condition (13),

$$(15) \quad -(\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 = 0.$$

Hence, we can introduce two complex valued functions f and g by

$$(16) \quad f := \psi^1 - i\psi^2, \quad g := \frac{\psi^0}{\psi^1 - i\psi^2}.$$

Using these two functions, φ can be written as

$$(17) \quad \varphi(z, \bar{z}) = 2\text{Re} \int_{z_0}^z \left(e^{\mu_1 x^2} f g, \frac{1}{2} e^{\mu_2 x^2} f(1+g^2), \frac{i}{2} f(1-g^2) \right) dz.$$

¹From here on we mean a surface by an immersion.

$\varphi^{-1}\varphi_z$ is given by

$$(18) \quad \varphi^{-1}\varphi_z = fgE_0 + \frac{1}{2}f(1+g^2)E_1 + \frac{i}{2}f(1-g^2)E_3.$$

So, the first fundamental form² I is given in terms of f and g by

$$(19) \quad \begin{aligned} I &= 2\langle \varphi^{-1}\varphi_z, \varphi^{-1}\varphi_{\bar{z}} \rangle dzd\bar{z} \\ &= |f|^2(1-|g|^2)^2 dzd\bar{z}. \end{aligned}$$

The normal Gauß map is computed to be

$$\varphi^{-1} \cdot N = \frac{1}{1-|g|^2} \left((1+|g|^2)E_0 + 2\operatorname{Re}(g)E_1 + 2\operatorname{Im}(g)E_2 \right).$$

Let $\mathbb{D} = \{\zeta^1 E_1 + \zeta^2 E_2 \subset \mathbb{R}^2 : (\zeta^1)^2 + (\zeta^2)^2 < 1\}$. Under the stereographic projection from $-E_0$

$$\wp^+ : \mathbb{H}^2(-1) \longrightarrow \mathbb{D}; \quad \wp^+(u^0 E_0 + u^1 E_1 + u^2 E_2) = \frac{u^1}{1+u^0} E_1 + \frac{u^2}{1+u^0} E_2,$$

the map $\varphi^{-1} \cdot N$ is identified with the function g . If $\mathbb{H}^2(-1)$ is defined to be the hyperboloid of two sheets

$$\mathbb{H}^2(-1) = \{u^0 E_0 + u^1 E_1 + u^2 E_2 : -(u^0)^2 + (u^1)^2 + (u^2)^2 = -1\},$$

then $\wp^+ : \mathbb{H}^2(-1) \longrightarrow \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the extended complex plane $\mathbb{C} \cup \{\infty\}$. The function g is called the *projected normal Gauß map* of φ . It follows from (11) and (12) that

$$(20) \quad \psi_{\bar{z}}^i = \mu_{i+1} \overline{\psi^i} \psi^2, \quad i = 0, 1,$$

$$(21) \quad \psi_{\bar{z}}^2 = \mu_1 |\psi^0|^2 - \mu_2 |\psi^1|^2.$$

Using (20) and (21), we obtain

$$(22) \quad \frac{\partial f}{\partial \bar{z}} = -i|f|^2 \left\{ \mu_1 |g|^2 - \frac{\mu_2}{2}(1+\bar{g}^2) \right\},$$

$$(23) \quad \frac{\partial g}{\partial \bar{z}} = \frac{i}{2} \bar{f} \{ \mu_1 \bar{g}(1+g^2) - \mu_2 g(1+\bar{g}^2) \}.$$

As is seen in Section 1, $G(0,0) = \mathbb{E}_1^3$ and $G(c,c) = \mathbb{H}_1^3(-c^2)_+$ are the only cases of solvable Lie group $G(\mu_1, \mu_2)$ with constant sectional curvature. For $G(0,0) = \mathbb{E}_1^3$,

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}} = 0,$$

²It can be also obtained directly from (14).

that is, both f and g are holomorphic. From (17), we retrieve the Weierstraß representation formula for maximal spacelike surface $\varphi : \mathcal{D} \rightarrow \mathbb{E}_1^3$ given by

$$(24) \quad \varphi(z, \bar{z}) = 2\operatorname{Re} \int_{z_0}^z \left(fg, \frac{1}{2}f(1+g^2), \frac{i}{2}f(1-g^2) \right) dz$$

in terms of holomorphic data (g, f) . (24) was obtained independently by O. Kobayashi [6] and by L. McNertney [9]. For $G(c, c) = \mathbb{H}_1^3(-c^2)_+$,

$$(25) \quad \frac{\partial f}{\partial \bar{z}} = -ic|f|^2 \left\{ |g|^2 - \frac{1}{2}(1 + \bar{g}^2) \right\},$$

$$(26) \quad \frac{\partial g}{\partial \bar{z}} = \frac{ic}{2} \bar{f}(\bar{g} - g)(1 - |g|^2).$$

Then the Weierstraß representation formula (17) with $\mu_1 = \mu_2 = c$ gives rise to maximal spacelike surfaces in $\mathbb{H}_1^3(-c^2)_+$. If g is holomorphic, it follows from (26) that $g = \bar{g}$ or $|g|^2 = 1$. If $|g|^2 = 1$ then we see from (19) that $I = 0$. If $g = \bar{g}$ then g is real. This means that $\psi^2 = 0$ (see (16)) and from the conformality condition (15) we get $(\psi^0)^2 = (\psi^1)^2$. But along with $\psi^2 = 0$ this also leads to $I = 0$. Hence the projected normal Gauß map of maximal spacelike surfaces in $\mathbb{H}_1^3(-c^2)_+$ cannot be holomorphic.

It follows from (22) and (23) that the projected normal Gauß map g satisfies the partial differential equation:

$$(27) \quad g_{z\bar{z}} - \frac{(\mu_1^2 - \mu_2^2)g(1+g^2)(1-\bar{g}^2)|g_{\bar{z}}|^2}{[\mu_1g(1+\bar{g}^2) - \mu_2\bar{g}(1+g^2)][\mu_1\bar{g}(1+g^2) - \mu_2g(1+\bar{g}^2)]} - \frac{2\mu_1|g|^2 - \mu_2(1+\bar{g}^2)}{\mu_1\bar{g}(1+g^2) - \mu_2g(1+\bar{g}^2)}g_zg_{\bar{z}} = 0.$$

The equation (27) is not the harmonic map equation for the projected normal Gauß map g in general. The following theorem tells under what conditions it becomes the harmonic map equation for g .

Theorem 7. *The partial differential equation (27) is the harmonic map equation for g if and only if $\mu_1^2 = \mu_2^2$. If $\mu_1 = \mu_2 \neq 0$, then (27) is simplified to*

$$(28) \quad g_{z\bar{z}} + \frac{1 + \bar{g}^2 - 2|g|^2}{(\bar{g} - g)(1 - |g|^2)}g_zg_{\bar{z}} = 0.$$

This equation is the harmonic map equation for a map $g : \mathfrak{D}(z, \bar{z}) \rightarrow \left(\hat{\mathbb{C}}(w, \bar{w}), \frac{2dw d\bar{w}}{|(\bar{w}-w)(1-|w|^2)|} \right)$. If $\mu_1 = -\mu_2$, then (27) is simplified to

$$(29) \quad g_{z\bar{z}} - \frac{1 + \bar{g}^2 + 2|g|^2}{(g + \bar{g})(1 + |g|^2)} g_z g_{\bar{z}} = 0.$$

This equation is the harmonic map equation for a map $g : \mathfrak{D}(z, \bar{z}) \rightarrow \left(\hat{\mathbb{C}}(w, \bar{w}), \frac{2dw d\bar{w}}{|(w+\bar{w})(1+|w|^2)|} \right)$.

Proof. The tension field $\tau(g)$ of g is given by ([1], [15])

$$(30) \quad \tau(g) = 4\lambda^{-2}(g_{z\bar{z}} + \Gamma_{ww}^w g_z g_{\bar{z}}),$$

where λ is a parameter of conformality. Here, Γ_{ww}^w denotes the Christoffel symbols of $\hat{\mathbb{C}}(w, \bar{w})$. Comparing the equations (27) and $\tau(g) = 0$, we see that (27) is a harmonic map equation if and only if $\mu_1^2 = \mu_2^2$. In order to find a suitable metric on $\hat{\mathbb{C}}(w, \bar{w})$ with which (27) is a harmonic map equation, one simply needs to solve the first order partial differential equations

$$\Gamma_{ww}^w = \begin{cases} \frac{1 + \bar{w}^2 - 2|w|^2}{(\bar{w} - w)(1 - |w|^2)} & \text{if } \mu_1 = \mu_2 \neq 0, \\ -\frac{1 + \bar{w}^2 + 2|w|^2}{(w + \bar{w})(1 + |w|^2)} & \text{if } \mu_1 = -\mu_2. \end{cases}$$

The solutions are

$$(g_{w\bar{w}}) = \begin{cases} \begin{pmatrix} 0 & \frac{1}{(\bar{w}-w)(1-|w|^2)} \\ \frac{1}{(\bar{w}-w)(1-|w|^2)} & 0 \end{pmatrix} & \text{if } \mu_1 = \mu_2 \neq 0, \\ \begin{pmatrix} 0 & \frac{1}{(w+\bar{w})(1+|w|^2)} \\ \frac{1}{(w+\bar{w})(1+|w|^2)} & 0 \end{pmatrix} & \text{if } \mu_1 = -\mu_2, \end{cases}$$

respectively. \square

Remark 2. It is well-known that the projected Gauß map g of a maximal spacelike surface in $G(0, 0) = \mathbb{E}_1^3$ satisfies the Laplace-Beltrami equation

$$\Delta g = 4\lambda^{-2} g_{z\bar{z}} = 0.$$

Remark 3. Theorem 7 tells us that Minkowski 3-space $G(0, 0) = \mathbb{E}_1^3$, anti-de Sitter 3-space $G(c, c) = \mathbb{H}_1^3(-c^2)_+$, and $G(c, -c)$ are the only homogeneous 3-spacetimes among $G(\mu_1, \mu_2)$ in which the projected normal Gauß map of a maximal spacelike surface is harmonic.

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