QUANTUM MECHANICS AS A GAUGE THEORY

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Abstract

We propose an alternative approach to gauge theoretical treatment of quantum mechanics by lifting quantum state functions to the holomorphic tangent bundle $T^+(\mathbb{C})$.

Introduction

In usual sense, quantum mechanics can be treated as a gauge theory by considering quantum state functions as sections of a complex line bundle over Minkowski spacetime $\mathbb{R}^{3+1}$. In this paper, we propose an alternative approach to a gauge theoretic treatment of quantum mechanics. A quantum state function $\psi : \mathbb{R}^{3+1} \longrightarrow \mathbb{C}$ may be lifted to a vector field (called a *lifted state*) to the holomorphic tangent bundle $T^+(\mathbb{C})$, where we regard $\mathbb{C}$ as a Hermitian manifold. The vector field can be regarded as a holomorphic section of $T^+(\mathbb{C})$ parametrized by space-time coordinates. The probability density of a lifted state function is naturally defined by Hermitian metric on $\mathbb{C}$. It turns out that the probability density of a state function coincides with that of its lifted state. Furthermore the Hilbert space structure of state functions is solely determined

by the Hermitan structure defined on each fibre $T_p^+(\mathbb{C})$ of $T^+(\mathbb{C})$. This means that as observables a state and its lifted state are not distinguishable and we may study a quantum mechanical model with lifted states in terms of Hermitian differential geometry, consistently with the standard quantum mechanics. An important application of the lifted quantum mechanics model is that when an external electromagnetic field is introduced, the covariant derivative of a lifted state function naturally gives rise to the new energy and momentum operators for a charged particle resulted from the presence of the external electromagnetic field. As a result we obtain new Schrödinger's equation that describes the motion of a charged particle under the influence of the external electromagnetic field.

1 A Parametrized Vector Field as a Quantum state Function

Here we regard $\mathbb{C}$ as a Hermitian manifold of complex dimension one with the hermitian metric

$$g = dz^\mu \otimes d\bar{z}^\mu. \tag{1}$$

Let $\mathbb{R}^{3+1}$ be the Minkowski 4-spacetime, which is $\mathbb{R}^4$ with coordinates (t, x^1, x^2, x^3) and Lorentz-Minkowski metric

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Hereafter we simply denote $\mathbb{R}^{3+1}$ by M. In quantum mechanics, a particle is described by a complex-valued wave function, a so-called *state function*, $\psi : M \longrightarrow \mathbb{C}$. The states ψ of a quantum mechanical system forms an infinite dimensional complex Hilbert space $\mathcal{H}$. In quantum mechanics the probability that a wave function ψ exists inside volume $V \subset M$ is given by

$$\int_V \psi^* \psi d^3 x,$$

where ψ^* denotes the complex conjugation of ψ. Since there is no reason for $\mathbb{C}$ to be the same complex vector space everywhere in the universe, rigorously ψ should be regarded as a section of a complex line bundle over M. When we do physics, we require sections (fields) to be nowhere vanishing so the vector bundle is indeed a trvial bundle over M, i.e. $M \times \mathbb{C}$. This kind of rigorous treatment of state functions is needed to study gauge theory and geometric quantization.

On the other hand, let $\phi : \mathbb{C} \longrightarrow T(\mathbb{C})$ be a vector field, where $T(\mathbb{C}) = \bigcup_{p \in \mathbb{C}} T_p(\mathbb{C})$ is the tangent bundle[1] of $\mathbb{C}$. The composite function $\psi_\phi := \phi \circ \psi : M \longrightarrow T(\mathbb{C})$ is a lift of ψ to $T(\mathbb{C})$ since any vector field is a section of the tangent bundle $T(\mathbb{C})$. Here we propose to study quantum mechanics by considering the lifts as state functions. The lifts can be regarded as vector fields, i.e. sections of tangent bundle, parametrized by spacetime coordinates. This way we can directly connect the Hilbert space structure on the space of states and the Hermitian metric on $\mathbb{C}$ i.e., in mathematical point of view, extending the notion of states as the lifts may allow us to study quantum mechanics not only in

terms of functional analysis (as theory of Hilbert spaces) but also in terms of differential geometry (as a gauge theory).

Definition 1. The probability of getting a particle described by a wave function ψ inside volume V is called the *expectation*[2] of ψ inside V.

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[1]Since each fibre $T_p(\mathbb{C})$ is a one-dimensional complex vector space, $T(\mathbb{C})$ is a complex line bundle.

[2]This should not confused with the expectation value or expected value in probability and statistics.

Definition 2. Let $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})$ be a state[3]. The *expectation* of ψ' inside volume V is defined by

$$\int_V g(\psi', \psi') d^3x, \tag{2}$$

where g is the Hermitian metric (1) on $\mathbb{C}$.

Clearly there are infinitely many choices of the lifts of ψ. Among them we are interested in a particular lift. In order to discuss that, let $\phi : \mathbb{C} \longrightarrow T(\mathbb{C})$ be a vector field defined in terms of real coordinates by

$$\phi(x, y) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{3}$$

In terms of complex variables, (3) is written as

$$\phi(z, \bar{z}) = z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}, \tag{4}$$

where ϕ is viewed as a map from $\mathbb{C}$ into the *complexified tangent bundle* of $\mathbb{C}$, $\phi : \mathbb{C} \longrightarrow T(\mathbb{C})^{\mathbb{C}} := \bigcup_{p\in\mathbb{C}} T_p(\mathbb{C})^{\mathbb{C}}$. Note that $T(\mathbb{C})^{\mathbb{C}} = T^+(\mathbb{C}) \oplus T^-(\mathbb{C})$ where $T^+(\mathbb{C}) = \bigcup_{p\in\mathbb{C}} T_p^+(\mathbb{C})$ and $T^-(\mathbb{C}) = \bigcup_{p\in\mathbb{C}} T_p^-(\mathbb{C})$ are, respectively, holomorphic and anti-holomorphic tangent bundles of $\mathbb{C}$. It should be noted that the holomorphic tangent bundles are holomorphic vector bundles.

Definition 3. Let E and M are complex manifolds and $\pi : E \longrightarrow M$ a holomorphic onto map. E is said to be a holomorphic vector bundle if

1. The typical fibre is $\mathbb{C}^n$ and the structure group is $\mathrm{GL}(n, \mathbb{C})$;

2. The local trivialization $\phi_\alpha : U_\alpha \times \mathbb{C}^n \longrightarrow \pi^{-1}(U_\alpha)$ is a biholomorphic map;

3. The transition map $h_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(n, \mathbb{C})$ is a holomorphic map.

Now,

$$\begin{aligned} \psi_\phi(\mathbf{r}, t) &:= \phi \circ \psi(\mathbf{r}, t) \\ &= \psi(\mathbf{r}, t)\left(\frac{\partial}{\partial z}\right)_{\psi(\mathbf{r},t)} + \bar{\psi}(\mathbf{r}, t)\left(\frac{\partial}{\partial \bar{z}}\right)_{\psi(\mathbf{r},t)} \in T(\mathbb{C})^{\mathbb{C}}. \end{aligned}$$

Recalling that $g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}}\right) = 0$ and $g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \frac{1}{2}$, we obtain

$$\int_V g(\psi_\phi, \psi_\phi) d^3 x = \int_V \psi\psi^* d^3 x.$$

Thus we have the following proposition holds:

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[3]Not every map $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})$ is regarded as a state function. This will be clarified in the following discussion.

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Now,

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\end{aligned}$$

Recalling that $g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}}\right) = 0$ and $g\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \frac{1}{2}$, we obtain

$$\int_V g(\psi_\phi, \psi_\phi) d^3 x = \int_V \psi \psi^* d^3 x.$$

Thus we have the following proposition holds:

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[3]Not every map $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})$ is regarded as a state function. This will be clarified in the following discussion.

Proposition 4. *Any state function* $\psi : M \longrightarrow \mathbb{C}$ *can be lifted to* $\psi' : M \longrightarrow T(\mathbb{C})^\mathbb{C}$ *such that*

$$\int_V g(\psi', \psi') d^3 x = \int_V \psi \psi^* d^3 x. \tag{5}$$

Physically the state functions ψ themselves are not observables but the distributions $|\psi|^2$ are. So the probabilities $\int_V |\psi|^2 d^3 x$ are also observables. Hence as long as the both state functions and their lifts have the same probabilities we may study quantum mechanics with the lifted state functions, consistently with standard quantum mechanics.

Definition 5. A map $\psi' : M \longrightarrow T(\mathbb{C})^\mathbb{C}$ is called a *lifted (quantum) state function* if

$$\int_V g(\psi', \psi') d^3 x = \int_V (\pi \circ \psi')(\pi \circ \psi')^* d^3 x. \tag{6}$$

Example 6. The map $\psi' : M \longrightarrow T(\mathbb{C})^\mathbb{C}$ given by

$$\psi'(r, t) = A e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \frac{\partial}{\partial z} + \bar{A} e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \frac{\partial}{\partial \bar{z}} \tag{7}$$

is a lifted state function. Note that $\psi := \pi \circ \psi' = A e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$ is a well-known *de Broglie wave*, a plane wave that describes the motion of a free particle with momentum $\mathbf{p} = \mathbf{k}\hbar$, in quantum mechanics [Greiner]. Also note that $\psi' = \psi_\phi$ where ϕ is the vector field given in (4).

2   The Holomorphic Tangent Bundle $T^+(\mathbb{C})$ and Hermitian Connection

From now on we will only consider a fixed vector field ϕ given in (4). Denote by ϕ^+ and ϕ^- the holomorphic and the anti-holomorphic parts, respectively. Since $\phi^- = \overline{\phi^+}$, without loss of generality we may only consider the lifts $\psi_{\phi^+} : \mathcal{M} \longrightarrow T^+(\mathbb{C})$. One can define an inner product, called a *Hermitian structure*, on the holomorphic tangent bundle $T^+(\mathbb{C})$ induced by the Hermitian metric g in (1):

Definition 7. We mean a *Hermitian structure* by an inner product on a holomorphic vector bundle $\pi : E \longrightarrow M$ of a complex manifold M whose action at $p \in M$ is $h_p : \pi^{-1}(p) \times \pi^{-1}(p) \longrightarrow \mathbb{C}$ such that

1. $h_p(u, av + bw) = ah_p(u, v) + bh_p(u, w)$ for $u, v, w \in \pi^{-1}(p)$, $a, b \in \mathbb{C}$,

2. $h_p(u, v) = \overline{h_p(v, u)}$, $u, v \in \pi^{-1}(p)$,

3. $h_p(u, u) \geq 0$, $h_p(u, u) = 0$, if and only if $u = h_\alpha^{-1}(p, 0)$, where $h_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^n$ is a (biholomorphic) local trivialization.

4. $h(s_1, s_2)$ is a complex-valued smooth function on M for $s_1, s_2 \in \Gamma(M, E)$, where $\Gamma(M, E)$ denotes the set of sections of the holomorphic vector bundle $\pi : E \longrightarrow M$.

The following proposition is straightforward.

Proposition 8. *For each $p \in \mathbb{C}$, define $h_p : T_p^+(\mathbb{C}) \times T_p^+(\mathbb{C}) \longrightarrow \mathbb{C}$ by*

$$h_p(u, v) = g_p(u, \bar{v}) \text{ for } u, v \in T_p^+(\mathbb{C}).$$

Then h is a Hermitian structure on $T^+(\mathbb{C})$.

Definition 9. The expectation of ψ_ϕ inside volume $V \subset M$ is defined simply by

$$\int_V h(\psi_{\phi^+}, \psi_{\phi^+}) d^3x. \tag{8}$$

Remark 10. Note that

$$\int_V h(\psi_{\phi^+}, \psi_{\phi^+}) d^3x = \int_V g(\psi_\phi, \psi_\phi) d^3x = \int_V \psi\psi^* d^3x.$$

For an obvious reason, we would like to differentiate sections. If we cannot differentiate sections (fields), we cannot do physics. Let $E \longrightarrow M$ is a vector bundle and $s : E \longrightarrow M$ a section. Let $\gamma : (-\epsilon, \epsilon) \longrightarrow M$ be a path through $\gamma(0) = m$. The conventional definition of the rate of change of s in the direction tangent to γ at m is

$$\lim_{t \to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}.$$

However, this definition makes no sense at all, because $s(\gamma(t)) \in F_{\gamma(t)}$ and $s(\gamma(0)) \in F_{\gamma(0)}$, and we cannot perform the required subtraction $s(\gamma(t)) - s(\gamma(0))$. Hence we need to come up with an alternative way to differentiate sections. It turns out that there is no unique way to differentiate sections and one needs to make a choice of differentiation depending on one's purpose. Differentiation of sections of a bundle can be done by introducing the notion of a *connection*. Here we particularly discuss a Hermitian connection. Denote by $\Gamma(M, E)$ the set of all sections $s : M \longrightarrow E$. Also denote by $\mathcal{F}(M)^\mathbb{C}$ the set of complex-valued functions on M. Given a Hermitian structure h, we can define a connection which is compatible with h.

Definition 11. Given a Hermitian structure h, we mean a *Hermitian connection* ∇ by a linear map $\nabla : \Gamma(M, E) \longrightarrow \Gamma(M, E \otimes T^*M^\mathbb{C})$ such that

1. $\nabla(fs) = (df) \otimes s + f\nabla s$, $f \in \mathcal{F}(M)^\mathbb{C}$, $s \in \Gamma(M, E)$. This is called *Leibniz rule*.

2. $d[h(s_1, s_2)] = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$. Due to this condition, we say that the Hermitian connection ∇ is compatible with Hermitian structure h.

3. $\nabla s = Ds + \bar{D}s$, where Ds and $\bar{D}s$, respectively, are a $(1,0)$-form and a $(0,1)$-form. It is demanded that $\bar{D} = \bar{\partial}$, where $\bar{\partial}$ is the Dolbeault operator.

Regarding a Hermitian connection we have the following important property holds:

Theorem 12. *Let M be a Hermitian manifold. Given a holomorphic vector bundle $\pi : E \longrightarrow M$ and a Hermitian structure h, there exists a unique Hermitian connection.*

Definition 13. A set of sections $\{\hat{e}_1, \cdots, \hat{e}_k\}$ is called a *unitary frame* if

$$h(\hat{e}_\mu, \hat{e}_\nu) = \delta_{\mu\nu}. \tag{9}$$

Associated with a tangent bundle TM over a manifold M is a principal bundle called the *frame bundle* $LM = \bigcup_{p \in M} L_p M$, where $L_p M$ is the set of frames at $p \in M$. Note that the *unitary* frame bundle LM is not a holomorphic vector bundle because the structure group $U(n)$ is not a complex manifold. Let $\{\hat{e}_1, \cdots, \hat{e}_k\}$ be a unitary frame. Define the local *connection one-form*[4] $\omega = (\omega_\mu^\nu)$ by

$$\nabla \hat{e}_\mu = \omega_\mu^\nu \otimes \hat{e}_\nu. \tag{10}$$

By a straightforward calculation, we obtain

Proposition 14.
$$\nabla^2 \hat{e}_\mu = \nabla \nabla \hat{e}_\mu = F_\mu^\nu \hat{e}_\nu. \tag{11}$$

The *curvature* of the Hermitian connection ∇ or physically *field strength* is defined by the 2-form

$$F = d\omega + \frac{1}{2}\omega \wedge \omega. \tag{12}$$

It follows from the definition of the Hermitian connection that:

Proposition 15. *Both the connection form ω and the curvature F are skew-Hermitian, i.e. ω, $F \in \mathfrak{u}(n)$ where $\mathfrak{u}(n)$ is the Lie algebra of the unitary group $U(n)$.*

In terms of the Lie bracket $[\quad , \quad]$ defined on $\mathfrak{u}(n)$, (12) can be written as

$$F = d\omega + [\omega, \omega] \tag{13}$$

By Theorem 12, there exists uniquely a Hermitian connection $\nabla : \Gamma(\mathbb{C}, T^+(\mathbb{C})) \longrightarrow \Gamma(\mathbb{C}, T^+(\mathbb{C}) \otimes T^*(\mathbb{C})^{\mathbb{C}})$. Let $\mathcal{H}_{\phi^+}$ be the set of all lifted state functions $\psi_{\phi^+} : \mathcal{M} \longrightarrow T^+(\mathbb{C})$. Endowed with the inner product induced by the Hermitian structure h, $\mathcal{H}_{\phi^+}$ becomes an infinite dimensional complex Hilbert space.

Now

$$
\begin{aligned}
\nabla \phi^+ &= \nabla \left(z \frac{\partial}{\partial z} \right) \\
&= dz \otimes \frac{\partial}{\partial z} + z \nabla \left(\frac{\partial}{\partial z} \right) \\
&= dz \otimes \frac{\partial}{\partial z} + \omega \otimes \frac{\partial}{\partial z} \\
&= (dz + \omega) \otimes \frac{\partial}{\partial z},
\end{aligned}
\tag{14}
$$

---

[4]Physicists usually call it the *gauge pontential*.

where $\omega \in \mathfrak{u}(1)$ is the connection one-form. Using the formula (14), we can define a covariant derivative $\nabla^{\phi^+} : \mathcal{H}_{\phi^+} \longrightarrow \Gamma(\mathbb{C}, T^+(\mathbb{C}) \otimes T^*(\mathbb{C})^{\mathbb{C}})$:

$$
\nabla^{\phi^+} \psi_{\phi^+} = (d\psi + \psi\omega) \otimes \frac{\partial}{\partial z}.
\tag{15}
$$

Using the formula (15), we can now differentiate our lifted state functions. This means we can do quantum mechanics with lifted state functions and that due to the nature of our connection in (15), we may treat quatum mechanics as a gauge theory as we will see in Section 4.

3   Sections of Frame Bundle LM and Gauge Transformations

In this section, we discuss only the case of complex line bundles for simplicity. It is also sufficient for us because our tangent bundle is essentially a complex line bundle. Let $\pi : L \longrightarrow M$ be a complex line bundle over a Hermitian manifold M of complex dimension one and ∇ a Hermitian connection of the vector bundle. Let $\hat{e}_\alpha$ be a unitary frame on a chart $U_\alpha \subseteq M$. Then there exist a connection one-form ω_α such that

$$
\nabla \hat{e}_\alpha = \omega_\alpha \otimes \hat{e}_\alpha.
\tag{16}
$$

Suppose that U_β is another chart of M such that $U_\alpha \cap U_\beta \neq \emptyset$. The transition map $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^\times$ can be defined by

$$
\hat{e}_\alpha = g_{\alpha\beta} \hat{e}_\beta.
\tag{17}
$$

Here $\mathbb{C}^{\times}$ denotes the multiplicative gorup of nonzero complex numbers. The transition map $g_{\alpha\beta}$ gives rise to the change of coordinates. Since $\hat{e}_{\alpha}$ and $\hat{e}_{\beta}$ are related by (17) on $U_{\alpha} \cap U_{\beta} \neq \emptyset$, we obtain

$$\begin{aligned} \nabla \hat{e}_{\alpha} &= \nabla(g_{\alpha\beta}\hat{e}_{\beta}) \\ &= (dg_{\alpha\beta}) \otimes \hat{e}_{\beta} + g_{\alpha\beta}\nabla\hat{e}_{\beta}. \end{aligned} \tag{18}$$

By (16) we have

$$\omega_{\alpha} \otimes \hat{e}_{\alpha} = (dg_{\alpha\beta} + g_{\alpha\beta}\omega_{\beta}) \otimes \hat{e}_{\beta} \tag{19}$$

or equivalently by (17)

$$\omega_{\alpha} = g_{\alpha\beta}^{-1}dg_{\alpha\beta} + \omega_{\beta}. \tag{20}$$

Note that $g_{\alpha\beta}^{-1}dg_{\alpha\beta} \in \mathfrak{u}(1)$. The formula (19) tells how the gauge potentials ω_{α} and ω_{β} are related. Physicists call (19) a *gauge transformation*. Just as a mathematical theory should not depend on a certain coordinate system, neither should a physical theory. It would be really awkward if we have two different physical theories regarding the same phenomenon here on Earth and on Alpha Centauri. For that reason, physicists require particle theory be *gauge invariant* (i.e. invariant under gauge transformations).

The converse is also true, namely if $\{\omega_{\alpha}\}$ is a collection of one-forms satisfying (20) on $U_{\alpha} \cap U_{\beta} \neq \emptyset$, then there exists a Hermitian connection ∇ such that $\nabla\hat{e}_{\alpha} = \omega_{\alpha} \otimes \hat{e}_{\alpha}$. First define $\nabla\hat{e}_{\alpha} = \omega_{\alpha} \otimes \hat{e}_{\alpha}$ for each section $\hat{e}_{\alpha} : U_{\alpha} \longrightarrow LM$. On $U_{\alpha} \cap U_{\beta} \neq \emptyset$, (18) holds and it must coincide with $\omega_{\alpha} \otimes \hat{e}_{\alpha}$. By (17) and (20)

$$\begin{aligned} \omega_{\alpha} \otimes \hat{e}_{\alpha} &= g_{\alpha\beta}^{-1}dg_{\alpha\beta} \otimes \hat{e}_{\alpha} + \omega_{\beta}\hat{e}_{\alpha} \\ &= dg_{\alpha\beta} \otimes (g_{\alpha\beta}^{-1}\hat{e}_{\alpha}) + \omega_{\beta}(g_{\alpha\beta}\hat{e}_{\beta}) \\ &= dg_{\alpha\beta} \otimes \hat{e}_{\beta} + g_{\alpha\beta}\nabla\hat{e}_{\beta}. \end{aligned}$$

Let $\xi \in \Gamma(M, LM)$ be an arbitrary section. Then $\xi|_{U_{\alpha}} = \xi_{\alpha}\hat{e}_{\alpha}$, where $\xi_{\alpha} : U_{\alpha} \longrightarrow \mathbb{C}$. By Leibniz rule

$$\begin{aligned} \nabla\xi|_{U_{\alpha}} &= d\xi_{\alpha} \otimes \hat{e}_{\alpha} + \xi_{\alpha}\nabla\hat{e}_{\alpha} \\ &= (d\xi_{\alpha} + \omega_{\alpha}\xi_{\alpha}) \otimes \hat{e}_{\alpha}. \end{aligned} \tag{21}$$

$\nabla\hat{e}_{\mu}^{\alpha}$ can be then extended to $\nabla\xi$ using (21).

Let F_{α} be the two-form[5]

$$F_{\alpha} = d\omega_{\alpha}$$

defined on U_{α}. Physically F_{α} is the field strength relative to the unitary frame field $\hat{e}_{\alpha} : U_{\alpha} \longrightarrow LM$. On $U_{\alpha} \cap U_{\beta} \neq \emptyset$, the gauge potentials ω_{α} and ω_{β} are related by the gauge tranformation (20). If F_{α} and F_{β} do not coincide on $U_{\alpha} \cap U_{\beta}$, it would be again a physically awkward situation. The following proposition tells that it will not happen.

Proposition 16. *Let F_α and F_β be the field strength relative to the unitary frame fields $\hat{e}_\alpha : U_\alpha \longrightarrow LM$ and $\hat{e}_\beta : U_\beta \longrightarrow LM$, respectively. If $U_\alpha \cap U_\beta \neq \emptyset$, then $F_\alpha = F_\beta$ on $U_\alpha \cap U_\beta$.*

Proof.

$$
\begin{aligned}
F_\alpha &= d\omega_\alpha \\
&= d(g_{\alpha\beta}^{-1} dg_{\alpha\beta} + \omega_\beta) \\
&= dg_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + g_{\alpha\beta}^{-1} d(dg_{\alpha\beta}) + d\omega_\beta \\
&= -g_{\alpha\beta}^{-1}(dg_{\alpha\beta})g_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + d\omega_\beta \\
&= d\omega_\beta = F_\beta,
\end{aligned}
$$

since $g_{\alpha\beta}g_{\alpha\beta}^{-1} = I$ and $d(dg_{\alpha\beta}) = 0$. $\qquad\square$

Physically what Proposition 16 says is that the field strength is invariant under the gauge transformation (19). The two-forms F_α and F_β agree on the intersection of two open sets U_α and U_β in the cover and hence define a global two-form. It is denoted by F and is called the curvature of ∇.

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[5] $F_\alpha \in \mathfrak{u}(1)$ and $\mathfrak{u}(1)$ is a commutative Lie algebra, so $[\omega_\alpha, \omega_\alpha] = 0$.

Remark 17. In a principal G-bundle, if the structure group G is a matrix Lie group, the gauge transformation is given by

$$
\omega_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1}\omega_\alpha g_{\alpha\beta}, \tag{22}
$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ is the transition map and the connection 1-forms (gauge potentials) ω_α takes values in $\mathfrak{g}$, the Lie algebra of G. The curvature (field strength) F is, of course, invariant under the gauge transformation (22) and is given by (13).

4 Quantum Mechanics of a Charged Particle in an Electromagnetic Field, as an Abelian Gauge Theory

In this section we consider a charged particle with charge e described by the state function $\psi : M \longrightarrow \mathbb{C}$. We simply write ∇^{ϕ^+} as ∇ because that will be the only covariant derivative we are going to consider hereafter. We also denote ψ_{ϕ^+} simply by ψ_ϕ.

Assume that $\omega \in \mathfrak{u}(1) = \mathfrak{so}(2)$. Then in terms of space-time coordinates (t, x^1, x^2, x^3), ω can be written as

$$
\omega = -\frac{ie}{\hbar}\rho dt - \frac{ie}{\hbar}A_\alpha dx^\alpha, \quad \alpha = 1, 2, 3
$$

where $\hbar$ is the Dirac constant[6]. The covariant derivative (15) then becomes

$$\nabla\psi_\phi \;=\; (d\psi + \omega) \otimes \left(\frac{\partial}{\partial z}\right)_\psi$$

$$= \left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho\right)\psi \left(\frac{\partial}{\partial z}\right)_\psi \otimes dt + \left(\frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha\right)\psi \left(\frac{\partial}{\partial z}\right)_\psi \otimes dx^\alpha \tag{23}$$

Define

$$\nabla_0 \;:=\; \left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho\right)\frac{\partial}{\partial z},$$

$$\nabla_\alpha \;:=\; \left(\frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha\right)\frac{\partial}{\partial z}, \quad \alpha = 1, 2, 3.$$

Definition 18. Let

$$D_j := \pi \circ \nabla_j, \quad j = 0, 1, 2, 3.$$

That is,

$$D_{0} = \frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho, \quad D_\alpha = \frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha.$$

Then D_j is called the *projected covariant derivative* of ∇_j. Equivalently, ∇_j is called the *lifted covariant derivative* of D_j.

---

[6] Also called the reduced Planck constant.

Remark 19. Interestingly, the complex Klein-Gordon field emerges rather naturally in the lifted quantum mechanics model, because the D_j are the gauge-invariant covariant derivatives of a charged complex Klein-Gordon field. If we consider ψ not as a quantum state function but as the fusion of two real fields representing a particle and its antiparticle, then we can obtain electrically charged Klein-Gordon fields by considering a relevant Lagrangian using the covariant derivatives D_j. See sections 3.9 and 3.10 of [Felsager] for details.

Now we discuss what the covariant derivatives (23) really mean. The Hamiltonian of a particle in quantum mechanics is given by

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \tag{24}$$

where $\mathbf{r}$ is the position operator and $\mathbf{p}$ is the momentum operator given by

$$p_\alpha = -i\hbar \frac{\partial}{\partial x^\alpha}. \tag{25}$$

In quantum mechanics, a state ψ evolves in time according to Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi. \tag{26}$$

Multiplying (23) by $-i\hbar$, we obtain

$$-i\hbar\nabla\psi_\phi = -i\hbar\left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho\right)\psi\left(\frac{\partial}{\partial z}\right)_\psi \otimes dt - i\hbar\left(\frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha\right)\psi\left(\frac{\partial}{\partial z}\right)_\psi \otimes dx^\alpha. \tag{27}$$

Intriguingly, (27) appears to be the momentum of lifted state ψ_ϕ. Set

$$
\begin{aligned}
\bar{E} &= i\hbar\frac{\partial}{\partial t} + e\rho \\
&= E + e\rho
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{p}_\alpha &= -i\hbar\frac{\partial}{\partial x^\alpha} - eA_\alpha \\
&= p_\alpha - eA_\alpha.
\end{aligned}
$$

Now we are naturally led to the following conjecture:

Conjecture 20. *Let*

$$
-Edt + p_\alpha dx^\alpha = -i\hbar\frac{\partial}{\partial t}dt + p_\alpha dx^\alpha
$$

be the momentum 4-vector of a particle with charge e when there is no presence of an electromagnetic field. If an electromagnetic field is introduced with electromagnetic potential $\rho dt + A_\alpha dx^\alpha$ as a background field, then the momentum 4-vector changes to

$$
-\bar{E}dt + \bar{p}_\alpha dx^\alpha = -(E + e\rho)dt + (p_\alpha - eA_\alpha)dx^\alpha. \tag{28}
$$

The Hamiltonian and Schrödinger's equation would then be replaced by

$$
\begin{aligned}
\bar{H}(\mathbf{r}, \bar{\mathbf{p}}) &= \frac{(\bar{\mathbf{p}})^2}{2m} + V(\mathbf{r}) \\
&= \frac{1}{2m}(p_\alpha - eA_\alpha)^2 + V(\mathbf{r})
\end{aligned}
$$

and

$$
\bar{E}\psi = \bar{H}\psi.
$$

The following theorem (Theorem (16.34) in [Frankel]) tells that our conjecture is indeed right.

Theorem 21. *Let $H = H(q, p, t)$ be the Hamiltonian for a charged particle, when no electromagnetic field is present. Let an electromagnetic field be introduced with electromagnetic potential $A = \rho dt + A_\alpha dx^\alpha$, $\alpha = 1, 2, 3$. Define a new canonical momentum variable p^* in $T^*\mathcal{M} \times \mathbb{R}$ by*

$$
p^*_\alpha := p_\alpha + eA_\alpha(t, q) \tag{29}
$$

and a new Hamiltonian

$$
H^*(q, p^*, t) := H(q, p, t) - e\rho(t, q) = H(q, p^* - eA, t) - e\rho(t, q). \tag{30}
$$

Then the particle of charge e satisfies new Hamiltonian equations

$$\frac{dq}{dt} = \frac{\partial H^*}{\partial p^*}$$

$$\frac{dp^*}{dt} = -\frac{\partial H^*}{\partial q} \tag{31}$$

$$\frac{dH^*}{dt} = \frac{\partial H^*}{\partial t}.$$

Proof. The theorem can be proved by comparing the solutions of the original system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

and the new system

$$\frac{dq}{dt} = \frac{\partial H^*}{\partial p^*}, \quad \frac{dp^*}{dt} = -\frac{\partial H^*}{\partial q}$$

as seen in [Frankel]. $\square$

Remark 22. Let λ and Ω denote the Poincaré 1-form and 2-form, respectively, given by

$$\lambda = -Hdt + p_\alpha dx^\alpha,$$
$$\Omega = d\lambda = d(-Hdt + p_\alpha dx^\alpha).$$

With new momenta $p_\alpha^* = p_\alpha + eA_\alpha$ and new Hamiltonian $H^* = H - e\rho$, the Poincaré 1-form can be defined by

$$\lambda^* = -H^*dt + p_\alpha^* dx^\alpha.$$

Accordingly the Poincaré 2-form is

$$\Omega^* = d\lambda^* = d(-H^*dt + p_\alpha^* dx^\alpha) = \Omega + eF,$$

where $F = dA$ is the electromagnetic field stregth. It can be shown that the Hamilton's equations can be simply written as

$$i_X\Omega^* = 0,$$

where $X = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dp}{dt}\frac{\partial}{\partial p}$.

If a particle described by ψ has charge e and there is an additional external electromagnetic field is present, by Theorem 21, the Hamiltonian (24) should be replaced by

$$H(\mathbf{r}, \mathbf{p}^*) = \frac{1}{2m}(p_\alpha^* - eA_\alpha)^2 + V(\mathbf{r}) - e\rho \tag{32}$$

and the canonical momenta p_α^* should be replaced by $p_\alpha^* = -i\hbar\frac{\partial}{\partial x^\alpha}$. Accordingly the Schrödinger's equation (26) becomes

$$i\hbar\left[\frac{\partial}{\partial t} - \left(\frac{ie}{\hbar}\right)\rho\right]\psi = -\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial x^\alpha} - \left(\frac{ie}{\hbar}\right)eA_\alpha\right]^2\psi + V\psi \tag{33}$$

or

$$i\hbar D_0\psi = -\frac{\hbar^2}{2m}D_\alpha D_\alpha \psi + V\psi. \tag{34}$$

Notice that this is exactly the same equation as the one we conjectured. Although $e\rho$ is regarded as a part of the Hamiltonian H^* in Theorem 21, we know that $e\rho$ can be also regarded as a part of energy operator as discussed in Conjecture 20.

Conclusion

In this paper, we discussed that by lifting quantum state functions to the holomorphic tangent bundle $T^+(\mathbb{C})$ we may be able to study quantum mechanics in terms of Hermitian differential geometry, consistently with the standard quantum mechanics. The proposed lifted quantum mechanics model also offers an alternative gauge theoretic treatment of quantum mechanics by considering a complex line bubble over $\mathbb{C}$ instead of the spacetime M. An advantage of the lifted quantum mechanics model is that when an external electromagnetic field is introduced, the covariant derivative of a lifted state function naturally gives rise to new energy and momentum operators for a charged particle resulted from the presence of the external electromagnetic field. As a result we obtain new Schrödinger's equation that describes the motion of a charged particle under the influence of the external electromagnetic field.

Further Questions for Future Research

1. In this paper, we considered quantum mechanics as abelian gauge theory by introducing electromagnetic field as a backround field. Can we study quantum mechanics as nonabelian gauge theory, for example SU(2)-gauge theory by introducing an $\mathfrak{su}(2)$-valued field? In that case, ψ needs to be considered as a spinor-valued map $\psi : M \longrightarrow \mathbb{C}^2$. If so, what are the possible physical applications?

2. Can we extend our results for a curved space-time? If so, we may be able to study the quantum mechanical motion of a charged particle in a charged (or a rotating) blackhole or a wormhole (J. Wheeler's problem. See pp.140–152 of [Baez-Muniain]).

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