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QUANTUM MECHANICS AS A GAUGE THEORY

by

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ABSTRACT

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In this thesis, we propose an alternative approach to gauge theoretical treatment of quantum mechanics by lifting quantum state functions to the holomorphic tangent bundle $T^+(\mathbb{C})$.

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Chapter 1

INTRODUCTION

It is well known that quantum mechanics can be treated as a gauge theory by considering quantum state functions as sections of a complex line bundle over Minkowski spacetime. In this thesis, we propose an alternative approach to a gauge theoretic treatment of quantum mechanics. A quantum state function $\psi : \mathbb{R}^{3+1} \longrightarrow \mathbb{C}$ can be lifted to a map (called a *lifted state*) to the holomorphic tangent bundle $T^+(\mathbb{C})$, where we regard $\mathbb{C}$ as a Hermitian manifold. The lifted state can be regarded as a holomorphic section (holomorphic vector field) of $T^+(\mathbb{C})$ parametrized by space-time coordinates. The probability density of a lifted state function is naturally defined by the standard Hermitian metric on $\mathbb{C}$. It turns out that the probability density of a state function coincides with that of its lifted state. Furthermore the Hilbert space structure of state functions is solely determined by the Hermitan structure defined on each fibre $T_p^+(\mathbb{C})$ of $T^+(\mathbb{C})$. This means that as observables, a state and its lifted state are not distinguishable, and we may study a quantum mechanical model with lifted states in terms of Hermitian differential geometry consistently with the standard quantum mechanics. Interestingly, in the proposed model, a charged complex Klein-Gordon field emerges naturally when ψ is considered as the fusion of two real fields representing a particle and its antiparticle. An important advantage of the lifted quantum mechanics model is that when an external electromagnetic field is introduced, the covariant derivative of a lifted state function naturally gives rise to new energy and momentum operators for a charged particle resulted from the presence of the external electromagnetic field. As a result we obtain a new Schrödinger's equation that describes the motion of a charged particle under the influence of the external electromagnetic field.

Chapters 2 and 3 are included in this thesis as preliminaries. In chapter 2, we briefly review basic Differential Geometry, especially Complex Manifolds, Hermitian Manifolds, and Fibre Bundles. Chapter 2 contains results mainly from [7] and [6]. In chapter 3, we briefly review some basics on Lagragian and Hamiltonian Mechanics. Most discussions in chapter 3 came from [3]. In chapter 4, we discuss main results from our research. As an application, we discuss quantum mechanics of a charged particle in an electromagnetic background field, as an abelian gauge theory (U(1)-gauge theory): In conclusion, we summarize our results and their significance. We also list some possibly interesting questions for future research, as applications of the proposed *lifted* quantum mechanics.

Chapter 2

SOME BASIC DIFFERENTIAL GEOMETRY

In this chapter, we briefly review some basic differential geometry. In particular, we review some basics in complex manifolds, Hermitian manifolds, and fibre bundles, that are necessary for our main discussions in chapter 4. The notions and results we mention in this chapter mostly come from [7] and [6].

2.1 Complex Manifolds

Definition 2.1.1. A map $f : \mathbb{C}^m \longrightarrow \mathbb{C}$ is said to be holomorphic if $f = f_1 + if_2$ satisfies the *Cauchy-Riemann equations*:

$$\frac{\partial f_1}{\partial x^\mu} = \frac{\partial f_2}{\partial y^\mu}, \; \frac{\partial f_2}{\partial x^\mu} = -\frac{\partial f_1}{\partial y^\mu} \tag{2.1}$$

where $z^\mu = x^\mu + iy^\mu$, $\mu = 1, 2, \cdots, m$ are the coordinates of $\mathbb{C}^m$. A map $(f^1, f^2, \cdots, f^n) : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is said to be holomorphic if each coordinate function f^α, $\alpha = 1, 2, \cdots, n$ is holomorphic.

A map $f : \mathbb{C}^m \longrightarrow \mathbb{C}$ is said to be anti-holomorphic if its conjugation $\bar{f}$ is holomorphic.

In $\mathbb{C}^m$, it is often more convenient to use complex coordinates $z^\mu, \bar{z}^\mu$ instead of real coordinates. By the chain rule one finds

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2} \left(\frac{\partial}{\partial x^\mu} - i \frac{\partial}{\partial y^\mu} \right), \; \frac{\partial}{\partial \bar{z}^\mu} = \frac{1}{2} \left(\frac{\partial}{\partial x^\mu} + i \frac{\partial}{\partial y^\mu} \right). \tag{2.2}$$

Proposition 2.1.1. *A map $f : \mathbb{C}^m \longrightarrow \mathbb{C}$ is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}^\mu} = 0$, $\mu = 1, 2, \cdots, m$. A map $f : \mathbb{C}^m \longrightarrow \mathbb{C}$ is anti-holomorphic if and only if $\frac{\partial f}{\partial z^\mu} = 0$, $\mu = 1, 2, \cdots, m$.*

Definition 2.1.2. A topological space M is a *complex manifold* if it satisfies the following conditions:

1. For any $p \in M$ there exists a neighborhood $U(p) \subset M$ which is homeomorphic to $\mathbb{C}^m$. Denote by ϕ the homeomorphism between $U(p)$ and $\mathbb{C}^m$. Then $U(p)$ and ϕ are called the *coordinate neighborhood* and the *coordinate map* of $p \in M$. The pair $(U(p), \phi)$ is called a *chart*. Clearly the space M is covered by such charts, and we can see that M needs to be even dimensional.

2. Given coordinate neighborhoods U_i and U_j with $U_i \cap U_j \neq \emptyset$, the change of coordinate maps $\phi_{ij} = \phi_j \circ \phi_i^{-1} : U_i \longrightarrow U_j$ and $\phi_{ji} = \phi_i \circ \phi_j^{-1} : U_j \longrightarrow U_i$ are holomorphic.

The number m is called the complex dimension of M and is denoted by $\dim_{\mathbb{C}} M = m$. The real dimension $2m$ is denoted either by $\dim_{\mathbb{R}} M = 2m$ or simply by $\dim M = 2m$.

Example 2.1.2. *$\mathbb{C}^m$ is a complex manifold of (complex) dimension m. It has a single chart $(\mathbb{C}^m, \iota_d)$.*

Example 2.1.3. *The 2-sphere S^2 is a complex manifold of (complex) dimension 1: Let $U_N = S^2 \setminus N$ and $U_S = S^2 \setminus S$, where $N = (0,0,1)$ and $S = (0,0,-1)$. Let $\varphi_N : U_N \longrightarrow \mathbb{C}$ and $\varphi_S : U_S \longrightarrow \mathbb{C}$ denote the stereographic projections from the north pole and the south pole, respectively. Then*[1]

$$\varphi_N(x^1, x^2, x^3) = \frac{x^1}{1-x^3} + i\frac{x^2}{1-x^3},$$
$$\varphi_S(x^1, x^2, x^3) = \frac{x^1}{1+x^3} - i\frac{x^2}{1+x^3}.$$

The inverse stereographic projection $\varphi_N^{-1} : \mathbb{C} \longrightarrow U_N$ is given by

$$\varphi_N^{-1}(z) = \left(\frac{2\mathrm{Re}z}{|z|^2+1}, \frac{2\mathrm{Im}z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right), \quad z \in \mathbb{C}.$$

Now the change of coordinate map $\varphi_S \circ \varphi_N^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ is given by

$$\varphi_S \circ \varphi_N^{-1}(z) = \frac{1}{z}, \quad z \in \mathbb{C},$$

which is a holomorphic map.

One may wonder if the sphere S^{2n} is a complex manifold for all n. First, the following proposition holds. This proposition can be proved straightforwardly using the stereographic projections from the north pole and the south pole as seen in Example 2.1.3.

Proposition 2.1.4. *If S^{2n} has the only one complex structure, then $n = 1$.*

In fact , it is known that S^2 has indeed only one complex structure. (See for instance [5].) Furthermore, it is also known that

Theorem 2.1.5. *S^{2n} has an almost complex structure[2] if and only if $n = 1$ or $n = 3$.*

---

[1] We changed the sign of the x^2 coordinate in $\varphi_S(x^1, x^2, x^3)$ in order to make the change of coordinate map holomorphic. Otherwise, it will be anti-holomorphic.

[2] The definition of almost complex structure follows below.

It is still an open problem whether S^6 admits a complex structure.

The differential operators in (2.2) form a basis of the tangent space $T_pM^{\mathbb{C}}$. Correspondingly the $2m$ 1-forms

$$dz^\mu = dx^\mu + idy^\mu, \ d\bar{z}^\mu = dx^\mu - idy^\mu \tag{2.3}$$

form the basis of the cotangent space $T_p^*M^{\mathbb{C}}$.

Let us define a linear map $J_p : T_pM \longrightarrow T_pM$ by

$$J_p\left(\frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial y^\mu}, \ J_p\left(\frac{\partial}{\partial y^\mu}\right) = -\frac{\partial}{\partial x^\mu}. \tag{2.4}$$

Then

$$J_p^2 = -\mathbf{1}_p \tag{2.5}$$

where $\mathbf{1}_p$ is the identity map on T_pM. As a matrix, J_p can be written as

$$J_p = \begin{pmatrix} O & -\mathbf{1} \\ \mathbf{1} & O \end{pmatrix}.$$

The map J is called the *almost complex structure* of a complex manifold M.

From equation (2.5), we see that the linear map J_p has two complex eigenvalues, $\pm i$. Denote by T_pM^+ and T_pM^- the eigenspaces corresponding to the eigenvalues i and $-i$, respectively; that is,

$$T_pM^\pm = \{Z \in T_pM^{\mathbb{C}} : J_pZ = \pm iZ\}.$$

Then

$$T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-.$$

Note that if $Z = Z^\mu \frac{\partial}{\partial z^\mu}$, then $J_pZ = iZ$, i.e. Z is an eigenvector of J_p corresponding to i. If $Z = Z^\mu \frac{\partial}{\partial \bar{z}^\mu}$, then $J_pZ = -iZ$, i.e. Z is an eigenvector of J_p corresponding to $-i$.

2.2   Hermitian Manifolds and Kähler Manifolds

Let M be a complex manifold with $\dim_{\mathbb{C}} M = m$ and g be a Riemannian metric of M as a differentiable manifold. Let $Z = X + iY$, $W = U + iV \in T_pM^{\mathbb{C}}$ and extend g so that

$$g_p(Z,W) = g_p(X,U) - g_p(Y,V) + i[g_p(X,V) + g_p(Y,U)].$$

The components of g with respect to the basis (2.2) are

$$g_{\mu\nu}(p) = g_p\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right),$$

$$g_{\mu\bar{\nu}}(p) = g_p\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right),$$

$$g_{\bar{\mu}\nu}(p) = g_p\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}\right),$$

$$g_{\bar{\mu}\bar{\nu}}(p) = g_p\left(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}\right).$$

These then satisfy

$$g_{\mu\nu} = g_{\nu\mu}, \; g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}}, \; g_{\bar{\mu}\nu} = g_{\nu\bar{\mu}}, \; \overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu}, \; \overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}}. \tag{2.6}$$

Definition 2.2.1. If a Riemannian metric g of a complex manifold M satisfies

$$g_p(J_pX, J_pY) = g_p(X, Y) \tag{2.7}$$

at each point $p \in M$ and for any $X, Y \in T_pM$, then g is said to be a *Hermitian metric*. The pair (M, g) is called a *Hermitian manifold*.

Remark 2.2.1. J_pX is always orthogonal to X with respect to a Hermitian metric, because

$$g_p(J_pX, X) = g_p(J_p^2X, J_pX) = -g_p(X, J_pX) = -g_p(J_pX, X).$$

Theorem 2.2.1. *A complex manifold always admits a Hermitian metric.*

Example 2.2.2. *Let $\mathbb{C}^n$ be the n-copies $\mathbb{C} \times \cdots \times \mathbb{C}$ of the complex field $\mathbb{C}$ with coordinates $(z^1, \cdots, z^n)$. Define a metric g on $\mathbb{C}^n$ by*

$$g = dz^\mu \otimes d\bar{z}^\mu. \tag{2.8}$$

Then $(\mathbb{C}^n, g)$ is a Hermitian manifold of (complex) dimension n.

Let (M, g) be a Hermitian manifold. Then

$$\begin{aligned}
g_{\mu\nu} &= g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) \\
&= g\left(J\left(\frac{\partial}{\partial z^\mu}\right), J\left(\frac{\partial}{\partial z^\nu}\right)\right) \\
&= -g\left(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right) \\
&= -g_{\mu\nu}.
\end{aligned}$$

Thus $g_{\mu\nu} = g_{\bar{\mu}\bar{\nu}} = 0$. Hence the Hermitian metric g takes the form

$$g = g_{\mu\bar{\nu}}dz^{\mu} \otimes d\bar{z}^{\nu} + g_{\bar{\mu}\nu}d\bar{z}^{\mu} \otimes dz^{\nu}.$$

Define a 2-form (a tensor field) Ω acting on $X, Y \in T_pM$ given by

$$\Omega_p(X,Y) = g_p(J_pX,Y). \tag{2.9}$$

Then Ω is antisymmetric, i.e. $\Omega(X,Y) = -\Omega(Y,X)$. Now,

$$\Omega(JX,JY) = g(J^2X,JY) = g(J^3X,J^2Y) = g(JX,Y) = \Omega(X,Y).$$

So Ω is invariant under the action of J. The tensor field (or 2-form) Ω is called the *Kähler form* of a Hermitian metric g.

Proposition 2.2.3. *The Kähler form Ω of a Hermitian metric g can be written as*

$$\Omega = ig_{\mu\bar{\nu}}dz^{\mu} \wedge d\bar{z}^{\nu}. \tag{2.10}$$

2.3 Fibre Bundles

Fibre bundles are, roughly speaking, topological spaces that are locally product spaces. More precisely they are defined as follows:

Definition 2.3.1. A *fibre bundle* is an object (E, M, F, π) consisting of

1. The *total space E*.

2. The *base space M* with an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$.

3. The fibre F and the projection map $\pi : E \longrightarrow M$.

The simplest case of a fibre bundle would be when the total space E is given by the product space $E = M \times F$. In this case, the bundle is called a *trivial bundle*. In general , the total space may just be too complicated for us to understand. So it would be nice if we could always find small parts of E that are simple enough for us to understand, say simply product spaces. For this reason we want the fibre bundle to have the following property as well: for each $U_\alpha \in \mathcal{U}$, there is a homeomorphism $h_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ which is called a *local trivialization*. For each $x \in M$, $F_x := \pi^{-1}(x)$ is homeomorphic to $\{x\} \times F$. F_x is called the fibre of x. Let $x \in U_\alpha \cap U_\beta \neq \emptyset$. The fibre F_x^α of x as a subset of $\pi^{-1}(U_\alpha)$ may not be the same as the fibre F_x^β of x as a subset of $\pi^{-1}(U_\beta)$; however, the two fibres F_x^α and F_x^β are

homeomorphic to each other. Denote by $h_{\alpha\beta}(x)$ the homeomorphism from F_x^α to F_x^β. In fact, for each $x \in M$, $h_{\alpha\beta}(x)$ is a homeomorphism from F to itself, i.e. $h_{\alpha\beta}(x) \in \mathrm{Aut}(F)$. The map $h_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(F)$ is called a *transition map*. For each $U_\alpha, U_\beta \in \mathcal{U}$ with $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \longrightarrow (U_\alpha \cap U_\beta) \times F$$

satisfies

$$h_\alpha \circ h_\beta^{-1}(x, f) = (x, h_{\alpha\beta}(x)(f))$$

for $x \in U_\alpha \cap U_\beta$ and $f \in F$.

Let M be a differentiable manifold of dimension n. Consider an atlas $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ along with coordinates $x_\alpha^1, \cdots, x_\alpha^n$ in U_α. For $x = (x_\alpha^1(x), \cdots, x_\alpha^n(x)) \in U_\alpha$, a tangent vector $v \in T_x M$ is given by

$$v = \sum_{j=1}^n v_\alpha^j \frac{\partial}{\partial x_\alpha^j}.$$

If $x \in U_\alpha \cap U_\beta$, then v is also written as

$$v = \sum_{j=1}^n v_\beta^j \frac{\partial}{\partial x_\beta^j}.$$

Here the change of coordinates is given by

$$v_\beta^j = v x_\beta^j = \sum_{k=1}^n v_\alpha^k \frac{\partial x_\beta^j}{\partial x_\alpha^k}.$$

For $x \in U_\alpha \cap U_\beta$ and $f = (f^1, \cdots, f^n) \in \mathbb{R}^n$, define

$$
\begin{aligned}
h_{\alpha\beta}(x)(f) &= \left(\sum_{k=1}^n \frac{\partial x_\beta^1}{\partial x_\alpha^k} f^k, \cdots, \sum_{k=1}^n \frac{\partial x_\beta^n}{\partial x_\alpha^k} f^k \right) \\
&= \begin{pmatrix} \frac{\partial x_\beta^1}{\partial x_\alpha^1} & \cdots & \frac{\partial x_\beta^1}{\partial x_\alpha^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_\beta^n}{\partial x_\alpha^1} & \cdots & \frac{\partial x_\beta^n}{\partial x_\alpha^n} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.
\end{aligned}
$$

Hence, $h_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Aut}(\mathbb{R}^n)$. The resulting bundle over M with fibre $F = \mathbb{R}^n$ is called the tangent bundle of M and is denoted by TM. Note that TM can also be regarded as the set of all tangent vectors of M, i.e.

$$TM = \bigcup_{x \in M} T_x M.$$

For each $x \in U_\alpha$, the fibre $\pi^{-1}(x)$ of $x \in M$ is $T_x\mathbb{R}^n \cong \{x\} \times \mathbb{R}^n$. The local trivialization map $h_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{R}^n$ is given by

$$h_\alpha(v) = (x, (v_\alpha^1, \cdots, v_\alpha^n)), \quad v \in T_xU_\alpha(= T_xM), \quad x \in U_\alpha.$$

A fibre bundle (E, M, F, π) is called a *vector bundle*[3] over M if each fibre F_x of $x \in M$ is a vector space. So tangent bundles are vector bundles. The tangent bundle TM is also a differentiable manifold of dimension $2n$ with local coordinates in $\pi^{-1}(U_\alpha)$, $(x_\alpha^1, \cdots, x_\alpha^n, v_\alpha^1, \cdots, v_\alpha^n)$. The Jacobian is given by

$$
\begin{aligned}
J(x_\beta^1, \cdots, x_\beta^n; x_\alpha^1, \cdots, x_\alpha^n) &= \frac{\partial(x_\beta^1, \cdots, x_\beta^n)}{\partial(x_\alpha^1, \cdots, x_\alpha^n)} \\
&= \begin{pmatrix} \frac{\partial x_\beta^1}{\partial x_\alpha^1} & \cdots & \frac{\partial x_\beta^1}{\partial x_\alpha^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_\beta^n}{\partial x_\alpha^1} & \cdots & \frac{\partial x_\beta^n}{\partial x_\alpha^n} \end{pmatrix} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(n, \mathbb{R}),
\end{aligned}
$$

where $\mathrm{GL}(n, \mathbb{R})$ denotes the *general linear group*, the group of linear isomorphisms from $\mathbb{R}^n$ to itself. In terms of matrices, $\mathrm{GL}(n, \mathbb{R})$ is the group of all $n \times n$ non-singular matrices with real entries. It should be noted that $\mathrm{GL}(n, \mathbb{R})$ is also a Lie group. Let $g_{\alpha\beta} := J(x_\beta^1, \cdots, x_\beta^n; x_\alpha^1, \cdots, x_\alpha^n)$. Then $g_{\alpha\beta}$ satisfies

$$
\begin{aligned}
g_{\alpha\alpha}(x) &= I_n, \\
g_{\beta\alpha}(x) &= g_{\alpha\beta}^{-1}(x), \\
g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) &= I_n, \quad x \in U_\alpha \cap U_\beta.
\end{aligned}
$$

The third property is called the *Čech cocycle condition*. The Lie group $\mathrm{GL}(n, \mathbb{R})$ acts on the fibre $\mathbb{R}^n$ by

$$h_{\alpha\beta}(x)(f) = g_{\alpha\beta}(x) \cdot f, \quad x \in U_\alpha \cap U_\beta, \quad f \in \mathbb{R}^n.$$

In general, if the transition map $h_{\alpha\beta}$ is the group action of a Lie group G on the fibre F, then the fibre bundle (E, M, F, π) is called a *G-bundle* and the Lie group G is called a *structure group*[4]. In particular, if the fibre itself is a Lie group, the G-bundle is called a *principal G-bundle*. Given a G-bundle (E, M, F, π) with structure group G, one can construct a principal G-bundle $P(E)$ associated with E by replacing F with G and regarding $g_{\alpha\beta}$ as transition maps. The tangent bundle TM can be regarded as a principal G-bundle with structure group

---

[3]This is a rough definition. We additionally require each $h_{\alpha\beta}(x) : F \longrightarrow F$ to be a linear isomorphism.

[4]Physicists often call structure groups gauge groups. But we actually mean *gauge group* by the group of all gauge transformations.

$\mathrm{GL}(n,\mathbb{R})$. The tangent bundle TM discussed here is a real vector bundle, since each fibre T_xM is a real vector space, i.e. a vector space over $\mathbb{R}$. Similarly, a vector bundle such that each fibre F_x is a complex vector space is called a complex vector bundle. In particular, if each fibre F_x is a one-dimensional complex vector space, the complex vector bundle is called a (complex) *line bundle*.

A *section s* of a vector bundle is like a vector field. It is a map $s : M \longrightarrow E$ such that $s(x) \in F_x$ or equivalently $\pi \circ s(x) = x$. From this definition clearly the section of a vector bundle is one-to-one.

Example 2.3.1. *Consider the trivial bundle $L = M \times \mathbb{C}$. Every section s looks like $s(x) = (x, f(x))$ for some function f.*

Example 2.3.2. *For a tangent bundle TM, sections are vector fields*

$$
\begin{aligned}
s : M &\longrightarrow TM \\
x &\longmapsto v_x \in T_xM.
\end{aligned}
$$

For the tangent bundle TS^2 over S^2, one can think of a section as a map $s : S^2 \longrightarrow TS^2$ such that $\langle s(x), x \rangle = 0$ for each $x \in S^2$.

The existence of sections and the triviality of bundles are closely related, as stated in the following theorem. This relationship is also very important in physics.

Theorem 2.3.3. *The bundles E and $P(E)$ are trivial if and only if $P(E)$ has a section.*

Regarding the triviality of bundles, it is also important to note that:

Theorem 2.3.4. *If the base space M is simply connected[5], then the fibre bundle (E, M, F, π) is a trivial bundle.*

---

[5]That is, the fundamental group of M is the trivial group, $\pi_1(M) = 0$.

Chapter 3

LAGRANGIAN AND HAMILTONIAN MECHANICS

In this chapter, we briefly review some basics on Lagrangian and Hamiltonian mechanics. The notions and results mentioned in this chapter mostly come from [3].

3.1 Lagrangian Mechanics

Let M^n be the configuration space of a dynamical system, and let $q^1, \cdots, q^n$ be local generalized coordinates. The *Lagragian* is a function of the generalized coordinates q and the generalized velocities $\dot{q} = \frac{dq}{dt}$, denoted by $L(q, \dot{q})$. It is important to note that q and $\dot{q}$ are $2n$-independent coordinates. So L can be considered as a real-valued function on the tangent bundle to M^n

$$L : TM^n \longrightarrow \mathbb{R}.$$

To discuss Hamiltonian formalism of classical mechanics later, we need to introduce variables p_i defined by

$$p_i(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}^i}. \tag{3.1}$$

The p_i's are called the *generalized momenta*. p is indeed a map

$$p : TM^n \longrightarrow T^*M^n,$$

where T^*M^n is the cotangent bundle to M^n, i.e. the space of cotangent vectors. p is bijective, and it can be made to an immersion by requiring that

$$\det \left(\frac{\partial p_i}{\partial \dot{q}^i} \right) \neq 0, \tag{3.2}$$

or equivalently

$$\det \left(\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} \right) \neq 0. \tag{3.3}$$

As a result, $p : TM^n \longrightarrow T^*M^n$ is a diffeomorphism. Usually there is no natural way to identify tangent vectors on M^n with cotangent vectors on M^n. But by introducing a Lagrangian function L, we manage to establish such an identification, $\dot{q}^j \frac{\partial}{\partial q^j} \xrightarrow{p} \frac{\partial L}{\partial \dot{q}^j} dq^j$. One must note that the identification changes with a change of L, i.e. a change of dynamics.

The ordered $2n$-tuple $(q^1, \cdots, q^n; p_1, \cdots, p_n)$ is called the local coordinates for T^*M^n. In mechanics, the cotangent bundle T^*M^n to the configuration space TM^n is called the *phase space* of the dynamical system.

Frequently the Lagrangian is of the form

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q) \tag{3.4}$$

where T is the kinetic energy and V is the potential energy. V is usually independent of $\dot{q}$ and T is frequently a positive definite symmetric quadratic form in the velocities

$$T(q, \dot{q}) = \frac{1}{2} g_{jk}(q) \dot{q}^j \dot{q}^k. \tag{3.5}$$

Thus the momentum p in (3.1) can be written as

$$p = \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial T}{\partial \dot{q}^i} = g_{ij}(q) \dot{q}^j. \tag{3.6}$$

If we consider $2T$ as a Riemannian metric on the configuration space M^n

$$\langle \dot{q}, \dot{q} \rangle = g_{ij}(q) \dot{q}^i \dot{q}^j,$$

then the kinetic energy represents half the length squared of the velocity vector.

Example 3.1.1. *Let $M = \mathbb{R}^2$. Then $TM = \mathbb{R}^4$. Consider two masses m_1 and m_2 moving in one dimension. Then the kinetic energy is given by*

$$T = \frac{1}{2} m_1 (\dot{q}_1)^2 + \frac{1}{2} m_2 (\dot{q}_2)^2$$

which is of the form in (3.6) by introducing the mass matrix

$$(g_{ij}) = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}.$$

3.2   The Poincaré 1-Form and 2-Form

We define the Poincaré 1-form λ by $\lambda = p_i dq^i$ in the phase space T^*M^n. It is well-defined on every cotangent bundle T^*M^n as seen in the following theorem.

Theorem 3.2.1. *There is a globally defined 1-form on every cotangent bundle T^*M^n, the* Poincaré 1-form λ. *In local coordinates (q, p) it is given by*

$$\lambda = p_i dq^i. \tag{3.7}$$

Having the Poincaré 1-form λ, we can also define the *Poincaré 2-form* by

$$\Omega = d\lambda = dp_i \wedge dq^i. \tag{3.8}$$

This form plays a very important role in Hamiltonian mechanics.

3.3 Hamiltonian Mechanics and Symplectic Geometry

We begin this section by introducing the interior product.

Definition 3.3.1. If $\mathbf{v}$ is a vector and α is a p-form, their *interior product $(p-1)$-form $i_{\mathbf{v}}\alpha$* is defined recursively by

$$i_{\mathbf{v}}\alpha = 0 \text{ if } \alpha \text{ is a 0-form}$$

$$i_{\mathbf{v}}\alpha = \alpha(\mathbf{v}) \text{ if } \alpha \text{ is a 1-form}$$

$i_{\mathbf{v}}\alpha(\mathbf{w}_2,\cdots,\mathbf{w}_p) = \alpha(\mathbf{v},\mathbf{w}_2,\cdots,\mathbf{w}_p)$ if α is a p-form.

Let us denote by $\bigwedge^p$ the vector space of p-forms. Then given a vector $\mathbf{v}$, the interior product $i_{\mathbf{v}}$ is a linear map $i_{\mathbf{v}} : \bigwedge^p \longrightarrow \bigwedge^{p-1}$. Since forms are linear, we have $i_{\mathbf{A}+\mathbf{B}} = i_{\mathbf{A}} + i_{\mathbf{B}}$ and $i_{a\mathbf{A}} = a i_{\mathbf{A}}$. $i_{\mathbf{v}}\alpha$ is sometimes written as $i(\mathbf{v})\alpha$.

$i_{\mathbf{v}} : \bigwedge^p \longrightarrow \bigwedge^{p-1}$ is an antiderivation:

Proposition 3.3.1. $i_{\mathbf{v}}(\alpha^p \wedge \beta^q) = [i_{\mathbf{v}}\alpha^p] \wedge \beta^q + (-1)^p \alpha^p \wedge [i_{\mathbf{v}}\beta^q]$, *where α^p denotes a p-form.*

Definition 3.3.2. A 2-form Ω on an even dimensional manifold M^{2n} is called *symplectic* if it satisfies

1. Ω is closed, i.e. $d\Omega = 0$.

2. Ω is nondegenerate, i.e. the linear transformation associating to a vector $\mathbf{X}$ the 1-form $i_{\mathbf{X}}\Omega$ is nonsingular.

(M^{2n}, Ω) is called a *symplectic manifold*.

Example 3.3.2. *The Poincaré 2-form (3.8) is a symplectic form. So every cotangent bundle is a symplectic manifold.*

Let $L = L(q,\dot{q})$ be a time-independent Lagrangian. We have a map $p : TM \longrightarrow T^*M$ given by $q^i = q^i$ and $p_i = \frac{\partial L}{\partial \dot{q}^i}$. As mentioned earlier we take this map as a diffeomorphism by requiring (3.2) or (3.3).

Definition 3.3.3. The *Hamiltonian function $H : T^*M \longrightarrow \mathbb{R}$* is defined by

$$H(q,p) = p_i \dot{q}^i - L(q,\dot{q}). \tag{3.9}$$

Proposition 3.3.3. *The Euler-Lagrange equations, $\frac{\partial L}{\partial q^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = 0$ in TM, translate to Hamilton's equations in the phase space T^*M*

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{3.10}$$

*Let **X** be a time-independent vector field on T^*M,*

$$\mathbf{X} = X^i\frac{\partial}{\partial q^i} + X^{i+n}\frac{\partial}{\partial p_i}.$$

*Then the integral curve of **X**, i.e. the solutions to*

$$\frac{dq^i}{dt} = X^i \text{ and } \frac{dp_i}{dt} = X^{i+n}$$

satisfy Hamilton's equations (3.10) if and only if the vector field X satisfies

$$i_{\mathbf{X}}\Omega = -dH. \tag{3.11}$$

The equation (3.11) is also referred to as Hamilton's equations and X is called a Hamiltonian vector field.

Let us now consider a time-dependent Hamiltonian $H = H(q, p, t)$. H is considered as a function on the extended phase space $T^*M \times \mathbb{R}$. Hamilton's equations are again (3.10) but note that

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial q^i}\right)\frac{dq^i}{dt} + \left(\frac{\partial H}{\partial p_i}\right)\frac{dp_i}{dt} + \frac{\partial H}{\partial t}.$$

We introduce new Poincaré forms

$$\lambda = p_i dq^i - H dt, \tag{3.12}$$
$$\Omega = d\lambda = dp_i \wedge dq^i - dH \wedge dt. \tag{3.13}$$

Let us consider a time-dependent vector field on $T^*M \times \mathbb{R}$ of the form

$$X = X^i\frac{\partial}{\partial q^i} + X^{i+n}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}.$$

Along the integral curves of X, we have

$$X = \left(\frac{dq^i}{dt}\right)\frac{\partial}{\partial q^i} + \left(\frac{dp_i}{dt}\right)\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}. \tag{3.14}$$

Proposition 3.3.4. *The integral curve X (3.14) satisfies Hamilton's equations (3.10) if and only if*

$$i_X\Omega = 0. \tag{3.15}$$

Such X is again called a Hamiltonian vector field and it is

$$X = \left(\frac{dH}{dp_i}\right)\frac{\partial}{\partial q^i} - \left(\frac{dH}{dq^i}\right)\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t}. \tag{3.16}$$

Chapter 4

QUANTUM MECHANICS AS A GAUGE THEORY

Now, we discuss our main results in this chapter.

4.1 A Parametrized Vector Field as a Quantum State Function

Here we regard $\mathbb{C}$ as a Hermitian manifold of complex dimension one with the Hermitian metric (2.8). Let $\mathbb{R}^{3+1}$ be the Minkowski 4-spacetime, which is $\mathbb{R}^4$ with coordinates (t, x^1, x^2, x^3) and Lorentz-Minkowski metric

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Hereafter, we simply denote $\mathbb{R}^{3+1}$ by $\mathcal{M}$. In quantum mechanics, a particle is described by a complex-valued wave function, a so-called *state function*, $\psi : \mathcal{M} \longrightarrow \mathbb{C}$. The states ψ of a quantum mechanical system form an infinite dimensional complex Hilbert space $\mathcal{H}$. In quantum mechanics the probability that a wave function ψ exists inside volume $V \subset \mathcal{M}$ is given by

$$\int_V \psi^* \psi d^3 x,$$

where ψ^* denotes the complex conjugation of ψ. Since there is no reason for $\mathbb{C}$ to be the same complex vector space everywhere in the universe, rigorously ψ should be regarded as a section of a complex line bundle over $\mathcal{M}$. When we do physics, we require sections (fields) to be nowhere vanishing, so the vector bundle is indeed a trivial bundle over $\mathcal{M}$, i.e. $\mathcal{M} \times \mathbb{C}$. This kind of rigorous treatment of state functions is needed to study gauge theory and geometric quantization.

On the other hand, let $\phi : \mathbb{C} \longrightarrow T(\mathbb{C})$ be a vector field, where $T(\mathbb{C}) = \bigcup_{p \in \mathbb{C}} T_p(\mathbb{C})$ is the tangent bundle[1] of $\mathbb{C}$. The composite function $\psi_\phi := \phi \circ \psi : \mathcal{M} \longrightarrow T(\mathbb{C})$ is a lift of ψ to $T(\mathbb{C})$ since any vector field is a section of the tangent bundle $T(\mathbb{C})$. Here we propose to study quantum mechanics by considering the lifts as state functions. The lifts can be regarded as vector fields, i.e. sections of tangent bundles, parametrized by spacetime coordinates. This way, we can directly connect the Hilbert space structure on the space of states and the Hermitian metric on $\mathbb{C}$, i.e. in a mathematical point of view, extending the

---

[1]Since each fibre $T_p(\mathbb{C})$ is a one-dimensional complex vector space, $T(\mathbb{C})$ is a complex line bundle.

notion of states as the lifts may allow us to study quantum mechanics not only in terms of functional analysis (as theory of Hilbert spaces), but also in terms of differential geometry (as a gauge theory).

Definition 4.1.1. The probability of getting a particle described by a wave function ψ inside volume V is called the *expectation*[2] of ψ inside V.

Definition 4.1.2. Let $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})$ be a state[3]. The *expectation* of ψ' inside volume V is defined by

$$\int_V g(\psi', \psi') d^3 x, \tag{4.1}$$

where g is the Hermitian metric (2.8) on $\mathbb{C}$.

Clearly there are infinitely many choices of the lifts of ψ. Among them we are interested in a particular lift. In order to discuss that, let $\phi : \mathbb{C} \longrightarrow T(\mathbb{C})$ be a vector field defined in terms of real coordinates by

$$\phi(x, y) = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \tag{4.2}$$

In terms of complex variables, (4.2) is written as

$$\phi(z, \bar{z}) = z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}}, \tag{4.3}$$

where ϕ is viewed as a map from $\mathbb{C}$ into the *complexified tangent bundle* of $\mathbb{C}$, $\phi : \mathbb{C} \longrightarrow T(\mathbb{C})^{\mathbb{C}} := \bigcup_{p \in \mathbb{C}} T_p(\mathbb{C})^{\mathbb{C}}$. Note that $T(\mathbb{C})^{\mathbb{C}} = T^+(\mathbb{C}) \oplus T^-(\mathbb{C})$ where $T^+(\mathbb{C}) = \bigcup_{p \in \mathbb{C}} T_p^+(\mathbb{C})$ and $T^-(\mathbb{C}) = \bigcup_{p \in \mathbb{C}} T_p^-(\mathbb{C})$ are, respectively, holomorphic and anti-holomorphic tangent bundles of $\mathbb{C}$. It should be noted that the holomorphic tangent bundles are holomorphic vector bundles.

Definition 4.1.3. Let E and M be complex manifolds and $\pi : E \longrightarrow M$ a holomorphic onto map. E is said to be a holomorphic vector bundle if

1. The typical fibre is $\mathbb{C}^n$ and the structure group is $\mathrm{GL}(n, \mathbb{C})$;

2. The local trivialization $\phi_\alpha : U_\alpha \times \mathbb{C}^n \longrightarrow \pi^{-1}(U_\alpha)$ is a biholomorphic map;

3. The transition map $h_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(n, \mathbb{C})$ is a holomorphic map.

---

[2]This should not confused with the expectation value or expected value in probability and statistics.
[3]Not every map $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})$ is regarded as a state function. This will be clarified in the following discussion.

Now,

$$
\begin{aligned}
\psi_\phi(\mathbf{r},t) \;&:=\; \phi \circ \psi(\mathbf{r},t) \\
&=\; \psi(\mathbf{r},t)\left(\frac{\partial}{\partial z}\right)_{\psi(\mathbf{r},t)} + \bar\psi(\mathbf{r},t)\left(\frac{\partial}{\partial \bar z}\right)_{\psi(\mathbf{r},t)} \in T(\mathbb{C})^{\mathbb{C}}.
\end{aligned}
$$

Recalling that $g\left(\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial \bar z},\frac{\partial}{\partial \bar z}\right) = 0$ and $g\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \bar z}\right) = \frac{1}{2}$, we obtain

$$
\int_V g(\psi_\phi,\psi_\phi)d^3x = \int_V \psi\psi^* d^3x.
$$

Thus we have that the following proposition holds:

Proposition 4.1.1. *Any state function $\psi : \mathcal{M} \longrightarrow \mathbb{C}$ can be lifted to $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})^{\mathbb{C}}$ such that*

$$
\int_V g(\psi',\psi')d^3x = \int_V \psi\psi^* d^3x. \tag{4.4}
$$

Physically the state functions ψ themselves are not observables but the distributions $|\psi|^2$ are. So the probabilities $\int_V |\psi|^2 d^3x$ are also observables. Hence, as long as both the state functions and their lifts have the same probabilities, we may study quantum mechanics with the lifted state functions consistently with standard quantum mechanics.

Definition 4.1.4. A map $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})^{\mathbb{C}}$ is called a *lifted (quantum) state function* if

$$
\int_V g(\psi',\psi')d^3x = \int_V (\pi \circ \psi')(\pi \circ \psi')^* d^3x. \tag{4.5}
$$

Example 4.1.2. *The map $\psi' : \mathcal{M} \longrightarrow T(\mathbb{C})^{\mathbb{C}}$ given by*

$$
\psi'(r,t) = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{\partial}{\partial z} + \bar A e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{\partial}{\partial \bar z} \tag{4.6}
$$

is a lifted state function. Note that $\psi := \pi \circ \psi' = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ is a well-known de Broglie wave, a plane wave that describes the motion of a free particle with momentum $\mathbf{p} = \mathbf{k}\hbar$, in quantum mechanics [4]. Also note that $\psi' = \psi_\phi$ where ϕ is the vector field given in (4.3).

4.2   The Holomorphic Tangent Bundle $T^+(\mathbb{C})$ and Hermitian Connection

From now on we will only consider a fixed vector field ϕ given in (4.3). Denote by ϕ^+ and ϕ^- the holomorphic and the anti-holomorphic parts, respectively. Since $\phi^- = \overline{\phi^+}$, without loss of generality we may only consider the lifts $\psi_{\phi^+} : \mathcal{M} \longrightarrow T^+(\mathbb{C})$. One can define an inner product, called a *Hermitian structure*, on the holomorphic tangent bundle $T^+(\mathbb{C})$ induced by the Hermitian metric g in (2.8):

Definition 4.2.1. We mean a *Hermitian structure* by an inner product on a holomorphic vector bundle $\pi : E \longrightarrow M$ of a complex manifold M whose action at $p \in M$ is $h_p : \pi^{-1}(p) \times \pi^{-1}(p) \longrightarrow \mathbb{C}$ such that

1. $h_p(u, av + bw) = ah_p(u, v) + bh_p(u, w)$ for $u, v, w \in \pi^{-1}(p)$, $a, b \in \mathbb{C}$,

2. $h_p(u, v) = \overline{h_p(v, u)}$, $u, v \in \pi^{-1}(p)$,

3. $h_p(u, u) \geq 0; h_p(u, u) = 0$, if and only if $u = h_\alpha^{-1}(p, 0)$, where $h_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^n$ is a (biholomorphic) local trivialization.

4. $h(s_1, s_2)$ is a complex-valued smooth function on M for $s_1, s_2 \in \Gamma(M, E)$, where $\Gamma(M, E)$ denotes the set of sections of the holomorphic vector bundle $\pi : E \longrightarrow M$.

The following proposition is straightforward.

Proposition 4.2.1. *For each $p \in \mathbb{C}$, define $h_p : T_p^+(\mathbb{C}) \times T_p^+(\mathbb{C}) \longrightarrow \mathbb{C}$ by*

$$h_p(u, v) = g_p(u, \bar{v}) \text{ for } u, v \in T_p^+(\mathbb{C}).$$

Then h is a Hermitian structure on $T^+(\mathbb{C})$.

Definition 4.2.2. The expectation of ψ_ϕ inside volume $V \subset \mathcal{M}$ is defined simply by

$$\int_V h(\psi_{\phi^+}, \psi_{\phi^+}) d^3x. \tag{4.7}$$

Remark 4.2.1. Note that

$$\int_V h(\psi_{\phi^+}, \psi_{\phi^+}) d^3x = \int_V g(\psi_\phi, \psi_\phi) d^3x = \int_V \psi \psi^* d^3x.$$

For an obvious reason, we would like to differentiate sections. If we cannot differentiate sections (fields), we cannot do physics. Let $E \longrightarrow M$ be a vector bundle and $s : E \longrightarrow M$ a section. Let $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ be a path through $\gamma(0) = m$. The conventional definition of the rate of change of s in the direction tangent to γ at m is

$$\lim_{t \to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}.$$

However, this definition makes no sense at all, because $s(\gamma(t)) \in F_{\gamma(t)}$ and $s(\gamma(0)) \in F_{\gamma(0)}$, and we cannot perform the required subtraction $s(\gamma(t)) - s(\gamma(0))$. Hence we need to come up with an alternative way to differentiate sections. It turns out that there is no unique way to differentiate sections and one needs to make a choice of differentiation depending on one's purpose. Differentiation of sections of a bundle can be done by introducing the notion of a

connection. Here we particularly discuss a Hermitian connection. Denote by $\Gamma(M,E)$ the set of all sections $s : M \longrightarrow E$. Also denote by $\mathcal{F}(M)^{\mathbb{C}}$ the set of complex-valued functions on M. Given a Hermitian structure h, we can define a connection which is compatible with h.

Definition 4.2.3. Given a Hermitian structure h, we mean a *Hermitian connection* ∇ by a linear map $\nabla : \Gamma(M,E) \longrightarrow \Gamma(M, E \otimes T^*M^{\mathbb{C}})$ such that

1. $\nabla(fs) = (df) \otimes s + f\nabla s$, $f \in \mathcal{F}(M)^{\mathbb{C}}$, $s \in \Gamma(M,E)$. This is called the *Leibniz rule.*

2. $d[h(s_1,s_2)] = h(\nabla s_1, s_2) + h(s_1, \nabla s_2)$. Due to this condition, we say that the Hermitian connection ∇ is compatible with Hermitian structure h.

3. $\nabla s = Ds + \bar{D}s$, where Ds and $\bar{D}s$, respectively, are a $(1,0)$-form and a $(0,1)$-form. It is demanded that $\bar{D} = \bar{\partial}$, where $\bar{\partial}$ is the Dolbeault operator.

Regarding a Hermitian connection, we have that the following important property holds:

Theorem 4.2.2. *Let M be a Hermitian manifold. Given a holomorphic vector bundle $\pi : E \longrightarrow M$ and a Hermitian structure h, there exists a unique Hermitian connection.*

Definition 4.2.4. A set of sections $\{\hat{e}_1, \cdots, \hat{e}_k\}$ is called a *unitary frame* if

$$h(\hat{e}_\mu, \hat{e}_\nu) = \delta_{\mu\nu}. \tag{4.8}$$

Associated with a tangent bundle TM over a manifold M is a principal bundle called the *frame bundle* $LM = \bigcup_{p \in M} L_p M$, where $L_p M$ is the set of frames at $p \in M$. Note that the *unitary* frame bundle LM is not a holomorphic vector bundle because the structure group $U(n)$ is not a complex manifold. Let $\{\hat{e}_1, \cdots, \hat{e}_k\}$ be a unitary frame. Define the local *connection one-form*[4] $\omega = (\omega_\mu^\nu)$ by

$$\nabla \hat{e}_\mu = \omega_\mu^\nu \otimes \hat{e}_\nu. \tag{4.9}$$

By a straightforward calculation, we obtain

Proposition 4.2.3.
$$\nabla^2 \hat{e}_\mu = \nabla \nabla \hat{e}_\mu = F_\mu^\nu \hat{e}_\nu. \tag{4.10}$$

The *curvature* of the Hermitian connection ∇ or, physically, the *field strength* is defined by the 2-form

$$F = d\omega + \frac{1}{2}\omega \wedge \omega. \tag{4.11}$$

It follows from the definition of the Hermitian connection that:

---

[4]Physicists usually call it the *gauge pontential*.

Proposition 4.2.4. *Both the connection form ω and the curvature F are skew-Hermitian, i.e. $\omega, F \in \mathfrak{u}(n)$ where $\mathfrak{u}(n)$ is the Lie algebra of the unitary group $U(n)$.*

In terms of the Lie bracket $[\quad,\quad]$ defined on $\mathfrak{u}(n)$, (4.11) can be written as

$$F = d\omega + [\omega, \omega]. \tag{4.12}$$

By Theorem 4.2.2, there exists uniquely a Hermitian connection $\nabla : \Gamma(\mathbb{C}, T^+(\mathbb{C})) \longrightarrow \Gamma(\mathbb{C}, T^+(\mathbb{C}) \otimes T^*(\mathbb{C})^{\mathbb{C}})$. Let $\mathcal{H}_{\phi^+}$ be the set of all lifted state functions $\psi_{\phi^+} : \mathcal{M} \longrightarrow T^+(\mathbb{C})$. Endowed with the inner product induced by the Hermitian structure h, $\mathcal{H}_{\phi^+}$ becomes an infinite dimensional complex Hilbert space.

Now

$$
\begin{aligned}
\nabla \phi^+ &= \nabla \left(z \frac{\partial}{\partial z} \right) \\
&= dz \otimes \frac{\partial}{\partial z} + z \nabla \left(\frac{\partial}{\partial z} \right) \\
&= dz \otimes \frac{\partial}{\partial z} + \omega \otimes \frac{\partial}{\partial z} \\
&= (dz + \omega) \otimes \frac{\partial}{\partial z},
\end{aligned} \tag{4.13}
$$

where $\omega \in \mathfrak{u}(1)$ is the connection one-form. Using the formula (4.13), we can define a covariant derivative $\nabla^{\phi^+} : \mathcal{H}_{\phi^+} \longrightarrow \Gamma(\mathbb{C}, T^+(\mathbb{C}) \otimes T^*(\mathbb{C})^{\mathbb{C}})$:

$$\nabla^{\phi^+} \psi_{\phi^+} = (d\psi + \psi\omega) \otimes \frac{\partial}{\partial z}. \tag{4.14}$$

Using formula (4.14), we can now differentiate our lifted state functions. This means we can do quantum mechanics with lifted state functions and due to the nature of our connection in (4.14), we may treat quantum mechanics as a gauge theory as we will see in Section 4.4.

4.3   Sections of Frame Bundle *LM* and Gauge Transformations

In this section, we discuss only the case of complex line bundles for simplicity. It is also sufficient for us because our tangent bundle is essentially a complex line bundle. Let $\pi : L \longrightarrow M$ be a complex line bundle over a Hermitian manifold M of complex dimension one and ∇ a Hermitian connection of the vector bundle. Let $\hat{e}_\alpha$ be a unitary frame on a chart $U_\alpha \subseteq M$. Then there exist a connection one-form ω_α such that

$$\nabla \hat{e}_\alpha = \omega_\alpha \otimes \hat{e}_\alpha. \tag{4.15}$$

Suppose that U_β is another chart of M such that $U_\alpha \cap U_\beta \neq \emptyset$. The transition map $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(1,\mathbb{C}) \cong \mathbb{C}^\times$ can be defined by

$$\hat{e}_\alpha = g_{\alpha\beta}\hat{e}_\beta. \tag{4.16}$$

Here $\mathbb{C}^\times$ denotes the multiplicative group of nonzero complex numbers. The transition map $g_{\alpha\beta}$ gives rise to the change of coordinates. Since $\hat{e}_\alpha$ and $\hat{e}_\beta$ are related by (4.16) on $U_\alpha \cap U_\beta \neq \emptyset$, we obtain

$$\begin{aligned}
\nabla\hat{e}_\alpha &= \nabla(g_{\alpha\beta}\hat{e}_\beta) \\
&= (dg_{\alpha\beta})\otimes\hat{e}_\beta + g_{\alpha\beta}\nabla\hat{e}_\beta. \tag{4.17}
\end{aligned}$$

By (4.15) we have

$$\omega_\alpha \otimes \hat{e}_\alpha = (dg_{\alpha\beta} + g_{\alpha\beta}\omega_\beta)\otimes\hat{e}_\beta, \tag{4.18}$$

or equivalently by (4.16),

$$\omega_\alpha = g_{\alpha\beta}^{-1}dg_{\alpha\beta} + \omega_\beta. \tag{4.19}$$

Note that $g_{\alpha\beta}^{-1}dg_{\alpha\beta} \in \mathfrak{u}(1)$. The formula (4.18) tells how the gauge potentials ω_α and ω_β are related. Physicists call (4.18) a *gauge transformation*. Just as a mathematical theory should not depend on a certain coordinate system, neither should a physical theory. It would be really awkward if we have two different physical theories regarding the same phenomenon here on Earth and on Alpha Centauri. For that reason, physicists require particle theory to be *gauge invariant* (i.e. invariant under gauge transformations).

The converse is also true, namely if $\{\omega_\alpha\}$ is a collection of one-forms satisfying (4.19) on $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a Hermitian connection ∇ such that $\nabla\hat{e}_\alpha = \omega_\alpha \otimes \hat{e}_\alpha$. First define $\nabla\hat{e}_\alpha = \omega_\alpha \otimes \hat{e}_\alpha$ for each section $\hat{e}_\alpha : U_\alpha \longrightarrow LM$. On $U_\alpha \cap U_\beta \neq \emptyset$, (4.17) holds and it must coincide with $\omega_\alpha \otimes \hat{e}_\alpha$. By (4.16) and (4.19),

$$\begin{aligned}
\omega_\alpha \otimes \hat{e}_\alpha &= g_{\alpha\beta}^{-1}dg_{\alpha\beta} \otimes \hat{e}_\alpha + \omega_\beta\hat{e}_\alpha \\
&= dg_{\alpha\beta} \otimes (g_{\alpha\beta}^{-1}\hat{e}_\alpha) + \omega_\beta(g_{\alpha\beta}\hat{e}_\beta) \\
&= dg_{\alpha\beta} \otimes \hat{e}_\beta + g_{\alpha\beta}\nabla\hat{e}_\beta.
\end{aligned}$$

Let $\xi \in \Gamma(M,LM)$ be an arbitrary section. Then $\xi|_{U_\alpha} = \xi_\alpha\hat{e}_\alpha$, where $\xi_\alpha : U_\alpha \longrightarrow \mathbb{C}$. By the Leibniz rule,

$$\begin{aligned}
\nabla\xi|_{U_\alpha} &= d\xi_\alpha \otimes \hat{e}_\alpha + \xi_\alpha\nabla\hat{e}_\alpha \\
&= (d\xi_\alpha + \omega_\alpha\xi_\alpha)\otimes\hat{e}_\alpha. \tag{4.20}
\end{aligned}$$

$\nabla\hat{e}_\mu^\alpha$ can be then extended to $\nabla\xi$ using (4.20).

Let F_α be the two-form[5]

$$F_\alpha = d\omega_\alpha$$

defined on U_α. Physically F_α is the field strength relative to the unitary frame field $\hat{e}_\alpha :$ $U_\alpha \longrightarrow LM$. On $U_\alpha \cap U_\beta \neq \emptyset$, the gauge potentials ω_α and ω_β are related by the gauge transformation (4.19). If F_α and F_β do not coincide on $U_\alpha \cap U_\beta$, it would again be a physically awkward situation. The following proposition tells us that it will not happen.

Proposition 4.3.1. *Let F_α and F_β be the field strength relative to the unitary frame fields $\hat{e}_\alpha : U_\alpha \longrightarrow LM$ and $\hat{e}_\beta : U_\beta \longrightarrow LM$, respectively. If $U_\alpha \cap U_\beta \neq \emptyset$, then $F_\alpha = F_\beta$ on $U_\alpha \cap U_\beta$.*

Proof.

$$
\begin{aligned}
F_\alpha &= d\omega_\alpha \\
&= d(g_{\alpha\beta}^{-1} dg_{\alpha\beta} + \omega_\beta) \\
&= dg_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + g_{\alpha\beta}^{-1} d(dg_{\alpha\beta}) + d\omega_\beta \\
&= -g_{\alpha\beta}^{-1}(dg_{\alpha\beta})g_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + d\omega_\beta \\
&= d\omega_\beta = F_\beta,
\end{aligned}
$$

since $g_{\alpha\beta}g_{\alpha\beta}^{-1} = I$ and $d(dg_{\alpha\beta}) = 0$. □

Physically what Proposition 4.3.1 says is that the field strength is invariant under the gauge transformation (4.18). The two-forms F_α and F_β agree on the intersection of two open sets U_α and U_β in the cover and hence define a global two-form. It is denoted by F and is called the curvature of ∇.

Remark 4.3.1. In a principal G-bundle, if the structure group G is a matrix Lie group, the gauge transformation is given by

$$\omega_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta}, \tag{4.21}$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ is the transition map and the connection 1-forms (gauge potentials) ω_α takes values in $\mathfrak{g}$, the Lie algebra of G. The curvature (field strength) F is, of course, invariant under the gauge transformation (4.21) and is given by (4.12).

---

[5]$F_\alpha \in \mathfrak{u}(1)$ and $\mathfrak{u}(1)$ is a commutative Lie algebra, so $[\omega_\alpha, \omega_\alpha] = 0$.

4.4 Quantum Mechanics of a Charged Particle in an Electromagnetic Field, as an Abelian Gauge Theory

In this section we consider a charged particle with charge e described by the state function $\psi : \mathcal{M} \longrightarrow \mathbb{C}$. We simply write ∇^{ϕ^+} as ∇ because this will be the only covariant derivative we are going to consider hereafter. We also denote $\psi_{\phi+}$ simply by ψ_ϕ.

Assume that $\omega \in \mathfrak{u}(1) = \mathfrak{so}(2)$. Then in terms of space-time coordinates (t, x^1, x^2, x^3), ω can be written as

$$\omega = -\frac{ie}{\hbar}\rho dt - \frac{ie}{\hbar}A_\alpha dx^\alpha, \quad \alpha = 1, 2, 3$$

where $\hbar$ is the Dirac constant[6]. The covariant derivative (4.14) then becomes

$$
\begin{aligned}
\nabla \psi_\phi &= (d\psi + \omega) \otimes \left(\frac{\partial}{\partial z} \right)_\psi \\
&= \left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho \right) \psi \left(\frac{\partial}{\partial z} \right)_\psi \otimes dt + \left(\frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha \right) \psi \left(\frac{\partial}{\partial z} \right)_\psi \otimes dx^\alpha. \quad (4.22)
\end{aligned}
$$

Define

$$
\begin{aligned}
\nabla_0 &:= \left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho \right) \frac{\partial}{\partial z}, \\
\nabla_\alpha &:= \left(\frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha \right) \frac{\partial}{\partial z}, \quad \alpha = 1, 2, 3.
\end{aligned}
$$

Definition 4.4.1. Let

$$D_j := \pi \circ \nabla_j, \quad j = 0, 1, 2, 3.$$

That is,

$$D_0 = \frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho, \quad D_\alpha = \frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha.$$

Then D_j is called the *projected covariant derivative* of ∇_j. Equivalently, ∇_j is called the *lifted covariant derivative* of D_j.

Remark 4.4.1. Interestingly, the complex Klein-Gordon field emerges rather naturally in the lifted quantum mechanics model, because the D_j are the gauge-invariant covariant derivatives of a charged complex Klein-Gordon field. If we consider ψ not as a quantum state function but as the fusion of two real fields representing a particle and its antiparticle, then we can obtain electrically charged Klein-Gordon fields by considering a relevant Lagrangian using the covariant derivatives D_j. See sections 3.9 and 3.10 of [2] for details.

---

[6]Also called the reduced Planck constant.

Now we discuss what the covariant derivatives (4.22) really mean. The Hamiltonian of a particle in quantum mechanics is given by

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \tag{4.23}$$

where $\mathbf{r}$ is the position operator and $\mathbf{p}$ is the momentum operator given by

$$p_\alpha = -i\hbar \frac{\partial}{\partial x^\alpha}. \tag{4.24}$$

In quantum mechanics, a state ψ evolves in time according to Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \tag{4.25}$$

Multiplying (4.22) by $-i\hbar$, we obtain

$$-i\hbar \nabla \psi_\phi = -i\hbar \left(\frac{\partial}{\partial t} - \frac{ie}{\hbar}\rho \right) \psi \left(\frac{\partial}{\partial z} \right)_\psi \otimes dt - i\hbar \left(\frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha \right) \psi \left(\frac{\partial}{\partial z} \right)_\psi \otimes dx^\alpha. \tag{4.26}$$

Intriguingly, (4.26) appears to be the momentum of lifted state ψ_ϕ. Set

$$\begin{aligned} \bar{E} &= i\hbar \frac{\partial}{\partial t} + e\rho \\ &= E + e\rho \end{aligned}$$

and

$$\begin{aligned} \bar{p}_\alpha &= -i\hbar \frac{\partial}{\partial x^\alpha} - eA_\alpha \\ &= p_\alpha - eA_\alpha. \end{aligned}$$

Now we are naturally led to the following conjecture:

Conjecture 4.4.1. *Let*

$$-E dt + p_\alpha dx^\alpha = -i\hbar \frac{\partial}{\partial t} dt + p_\alpha dx^\alpha$$

be the momentum 4-vector of a particle with charge e when there is no presence of an electromagnetic field. If an electromagnetic field is introduced with electromagnetic potential $\rho dt + A_\alpha dx^\alpha$ as a background field, then the momentum 4-vector changes to

$$-\bar{E} dt + \bar{p}_\alpha dx^\alpha = -(E + e\rho)dt + (p_\alpha - eA_\alpha)dx^\alpha. \tag{4.27}$$

The Hamiltonian and Schrödinger's equation would then be replaced by

$$\bar{H}(\mathbf{r},\bar{\mathbf{p}}) = \frac{(\bar{\mathbf{p}})^2}{2m} + V(\mathbf{r})$$

$$= \frac{1}{2m}(p_\alpha - eA_\alpha)^2 + V(\mathbf{r})$$

and

$$\bar{E}\psi = \bar{H}\psi.$$

The following theorem (Theorem (16.34) in [3]) tells that our conjecture is indeed right.

Theorem 4.4.2. *Let $H = H(q, p, t)$ be the Hamiltonian for a charged particle, when no electromagnetic field is present. Let an electromagnetic field be introduced with electromagnetic potential $A = \rho dt + A_\alpha dx^\alpha$, $\alpha = 1, 2, 3$. Define a new canonical momentum variable p^* in $T^*\mathcal{M} \times \mathbb{R}$ by*

$$p_\alpha^* := p_\alpha + eA_\alpha(t, q) \tag{4.28}$$

and a new Hamiltonian

$$H^*(q, p^*, t) := H(q, p, t) - e\rho(t, q) = H(q, p^* - eA, t) - e\rho(t, q). \tag{4.29}$$

Then the particle of charge e satisfies new Hamiltonian equations

$$\begin{aligned}
\frac{dq}{dt} &= \frac{\partial H^*}{\partial p^*} \\
\frac{dp^*}{dt} &= -\frac{\partial H^*}{\partial q} \\
\frac{dH^*}{dt} &= \frac{\partial H^*}{\partial t}.
\end{aligned} \tag{4.30}$$

Proof. The theorem can be proved by comparing the solutions of the original system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

and the new system

$$\frac{dq}{dt} = \frac{\partial H^*}{\partial p^*}, \quad \frac{dp^*}{dt} = -\frac{\partial H^*}{\partial q}$$

as seen in [3]. $\square$

Remark 4.4.2. Let λ and Ω denote the Poincaré 1-form and 2-form, respectively, given by

$$\begin{aligned}
\lambda &= -Hdt + p_\alpha dx^\alpha, \\
\Omega &= d\lambda = d(-Hdt + p_\alpha dx^\alpha).
\end{aligned}$$

With new momenta $p^*_\alpha = p_\alpha + eA_\alpha$ and new Hamiltonian $H^* = H - e\rho$, the Poincaré 1-form can be defined by

$$\lambda^* = -H^* dt + p^*_\alpha dx^\alpha.$$

Accordingly the Poincaré 2-form is

$$\Omega^* = d\lambda^* = d(-H^* dt + p^*_\alpha dx^\alpha) = \Omega + eF,$$

where $F = dA$ is the electromagnetic field stregth. It can be shown that Hamilton's equations can simply be written as

$$i_X \Omega^* = 0,$$

where $X = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dp}{dt}\frac{\partial}{\partial p}$.

If a particle described by ψ has charge e and there is an additional external electromagnetic field present, by Theorem 4.4.2, the Hamiltonian (4.23) should be replaced by

$$H(\mathbf{r}, \mathbf{p}^*) = \frac{1}{2m}(p^*_\alpha - eA_\alpha)^2 + V(\mathbf{r}) - e\rho \tag{4.31}$$

and the canonical momenta p^*_α should be replaced by $p^*_\alpha = -i\hbar\frac{\partial}{\partial x^\alpha}$. Accordingly Schrödinger's equation (4.25) becomes

$$i\hbar\left[\frac{\partial}{\partial t} - \left(\frac{ie}{\hbar}\right)\rho\right]\psi = -\frac{\hbar^2}{2m}\left[\frac{\partial}{\partial x^\alpha} - \left(\frac{ie}{\hbar}\right)eA_\alpha\right]^2\psi + V\psi \tag{4.32}$$

or

$$i\hbar D_0\psi = -\frac{\hbar^2}{2m}D_\alpha D_\alpha\psi + V\psi. \tag{4.33}$$

Notice that this is exactly the same equation as the one we conjectured. Although $e\rho$ is regarded as a part of the Hamiltonian H^* in Theorem 4.4.2, we know that $e\rho$ can be also regarded as a part of the energy operator as discussed in Conjecture 4.4.1.

Chapter 5

CONCLUSION

In this thesis, we discussed that by lifting quantum state functions to the holomorphic tangent bundle $T^+(\mathbb{C})$ we may be able to study quantum mechanics in terms of Hermitian differential geometry, consistent with standard quantum mechanics. The proposed lifted quantum mechanics model also offers an alternative gauge theoretic treatment of quantum mechanics by considering a complex line bundle over $\mathbb{C}$ instead of the spacetime $\mathcal{M}$. An advantage of the lifted quantum mechanics model is that when an external electromagnetic field is introduced, the covariant derivative of a lifted state function naturally gives rise to new energy and momentum operators for a charged particle resulted from the presence of the external electromagnetic field. As a result we obtain a new Schrödinger's equation that describes the motion of a charged particle under the influence of the external electromagnetic field.

The following questions may be considered for future research: 1. In this thesis, we considered quantum mechanics as an abelian gauge theory, as an application of *lifted* quantum mechanics, by introducing electromagnetic field as a backround field. Can we study similarly quantum mechanics as a nonabelian gauge theory, for example as an $\mathfrak{su}(2)$-valued field? In that case, ψ needs to be considered as a spinor-valued map $\psi : \mathcal{M} \longrightarrow \mathbb{C}^2$. If so, what are the possible physical applications? 2. Can we extend our results for a curved space-time? If so, we may be able to study the quantum mechanical motion of a charged particle in a charged (or a rotating) blackhole or a wormhole (J. Wheeler's problem. See pp.140–152 of [1]).

BIBLIOGRAPHY

[1] J. Baez and J.P. Muniain. *Gauge Theories, Knots and Gravity*. World Scientific, 1994.

[2] B. Felsager. *Geometry, Particles and Fields, 2nd Edition*. Odense University Press, 1983.

[3] T. Frankel. *The Geometry of Physics, An Introduction*. Cambridge University Press, 2001.

[4] W. Greiner. *Quantum Mechanics, An Introduction, 4th Edition*. Springer, 2001.

[5] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry*. Wiley, 1996.

[6] M. Murray. *Notes on Line Bundles*.
     Found at http://www.maths.adelaide.edu.au/michael.murray/line_bundles.pdf.

[7] M. Nakahara. *Geometry, Topology, and Physics, 2nd Edition*. Institute of Physics Publishing, 2003.

[8] C. Nash and S. Sen. *Topology and Geometry for Physicists*. Academic Press, 1987.