Minimal surfaces in a certain 3-dimensional homogeneous spacetime

Sungwook Lee

Abstract

The 2-parameter family of certain homogeneous Lorentzian 3-manifolds which includes Minkowski 3-space, de Sitter 3-space, and Minkowski motion group is considered. Each homogeneous Lorentzian 3-manifold in the 2-parameter family has a solvable Lie group structure with left invariant metric. A generalized integral representation formula which is the unification of representation formulas for minimal timelike surfaces in those homogeneous Lorentzian 3-manifolds is obtained. The normal Gauß map of minimal timelike surfaces in those homogeneous Lorentzian 3-manifolds and its harmonicity are discussed.

Keywords: de Sitter space, harmonic map, homogeneous manifold, Lorentz surface, Lorentzian manifold, Minkowski space, minimal surface, solvable Lie group, spacetime, timelike surface

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Introduction

In [5], the author considered the 2-parameter family of 3-dimensional homogeneous spacetimes $(\mathbb{R}^3(x^0, x^1, x^2), g_{(\mu_1,\mu_2)})$ with Lorentzian metric

$$g_{(\mu_1,\mu_2)} = -(dx^0)^2 + e^{-2\mu_1 x^0}(dx^1)^2 + e^{-2\mu_2 x^0}(dx^2)^2.$$

Every homogeneous Lorentzian manifold in this family can be represented as a solvable matrix Lie group with left invariant metric

$$G(\mu_1, \mu_2) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x^0 \\ 0 & e^{\mu_1 x^0} & 0 & x^1 \\ 0 & 0 & e^{\mu_2 x^0} & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; x^0, x^1, x^2 \in \mathbb{R} \right\}.$$

As special cases, this family of homogeneous Lorentzian 3-manifolds include Minkowski 3-space $\mathbb{E}^3_1$, de Sitter 3-space $\mathbb{S}^3_1(c^2)$ of constant sectional curvature c^2 as a warped product model, and $\mathbb{S}^2_1(c^2)\times\mathbb{E}^1$, the direct product of de Sitter 2-space $\mathbb{S}^2_1(c^2)$ of constant curvature c^2 and the real line $\mathbb{E}^1$. (In fact, Minkowski 3-space and de Sitter 3-space are the only homogeneous Lorentzian 3-manifolds in this family that have a constant sectional curvature.) These three spaces may be considered as Lorentzian counterparts of Euclidean 3-space $\mathbb{E}^3$, hyperbolic 3-space $\mathbb{H}^3(-c^2)$ of constant sectional curvature $-c^2$, and $\mathbb{H}^2(-c^2)\times\mathbb{E}^1$, the direct product of hyperbolic plane $\mathbb{H}^2(-c^2)$ of constant curvature $-c^2$ and the real line $\mathbb{E}^1$, respectively, of Thurston's eight model geometries [8]. In [5], the author obtained a generalized integral representation formula which is the unification of representation formulas for maximal spacelike surfaces in those homogeneous Lorentzian 3-manifolds. In particular, the generalized integral formula includes Weierstraß representation formula for maximal spacelike surfaces in Minkowski 3-space studied independently by O. Kobayashi [4] and L. McNertney [7], and Weierstraß representation formula for maximal spacelike surfaces in de Sitter 3-space.

In this paper, the author obtains a generalized integral representation formula which is the unification of representation formulas for minimal timelike surfaces in those homogeneous Lorentzian 3-manifolds. In particular, the generalized integral formula includes Weierstraß representation formula for minimal timelike surfaces in Minkowski 3-space ([3], [6]) and Weierstraß representation formula for minimal timelike surfaces in de Sitter 3-space. The harmonicity of the normal Gauß map of minimal timelike surfaces in $G(\mu_1,\mu_2)$ is also discussed. It is shown that Minkowski 3-space $G(0,0)$, de Sitter 3-space $G(c,c)$, and Minkowski motion group $G(c,-c)$ are the only homogeneous Lorentzian 3-manifolds among the 2-parameter family members $G(\mu_1,\mu_2)$ in which the (projected) normal Gauß map of minimal timelike surfaces is harmonic. The harmonic map equations for those cases are obtained.

1 Solvable Lie group

In this section, we study the two-parameter family of certain homogeneous Lorentzian 3-manifolds.

Let us consider the two-parameter family of homogeneous Lorentzian 3-manifolds

$$\left\{(\mathbb{R}^3(x^0,x^1,x^2),g_{(\mu_1,\mu_2)})\mid(\mu_1,\mu_2)\in\mathbb{R}^2\right\}, \tag{1.1}$$

where the metric $g_{(\mu_1,\mu_2)}$ is defined by

$$g_{(\mu_1,\mu_2)} := -(dx^0)^2 + e^{-2\mu_1 x^0}(dx^1)^2 + e^{-2\mu_2 x^0}(dx^2)^2. \tag{1.2}$$

Proposition 1.1 *Each homogeneous space $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is isometric to the following solvable matrix Lie group:*

$$G(\mu_1,\mu_2) = \left\{ \left(\begin{array}{cccc} 1 & 0 & 0 & x^0 \\ 0 & e^{\mu_1 x^0} & 0 & x^1 \\ 0 & 0 & e^{\mu_2 x^0} & x^2 \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| \; x^0, x^1, x^2 \in \mathbb{R} \right\}$$

with left invariant metric. The group operation on $G(\mu_1,\mu_2)$ is the ordinary matrix multiplication and the corresponding group operation on $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$ is given by

$$(x^0, x^1, x^2) \cdot (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2) = (x^0 + \tilde{x}^0, x^1 + e^{\mu_1 x^0}\tilde{x}^1, x^2 + e^{\mu_2 x^0}\tilde{x}^2).$$

Proof. For $\tilde{a} = (a^0, a^1, a^2) \in G(\mu_1,\mu_2)$, denote by $L_{\tilde{a}}$ the left translation by $\tilde{a}$. Then

$$L_{\tilde{a}}(x^0, x^1, x^2) = (x^0 + a^0, e^{\mu_1 a^0} x^1 + a^1, e^{\mu_2 a^0} x^2 + a^2)$$

and

$$\begin{aligned} L_{\tilde{a}}^* g_{(\mu_1,\mu_2)} =& - \{d(x^0 + a^0)\}^2 + e^{-2\mu_1(x^0+a^0)}\{d(e^{\mu_1 a^0} x^1 + a^1)\}^2 \\ &+ e^{-2\mu_2(x^0+a^0)}\{d(e^{\mu_2 a^0} x^2 + a^2)\}^2 \\ =& - (dx^0)^2 + e^{-2\mu_1 x^0}(dx^1)^2 + e^{-2\mu_2 x^0}(dx^2)^2. \end{aligned}$$

Q.E.D.

The Lie algebra $\mathfrak{g}(\mu_1,\mu_2)$ is given by

$$\mathfrak{g}(\mu_1,\mu_2) = \left\{ \left(\begin{array}{cccc} 0 & 0 & 0 & y^0 \\ 0 & \mu_1 y^0 & 0 & y^1 \\ 0 & 0 & \mu_2 y^0 & y^2 \\ 0 & 0 & 0 & 0 \end{array} \right) \middle| \; y^0, y^1, y^2 \in \mathbb{R} \right\}. \tag{1.3}$$

We take the following basis $\{E_0, E_1, E_2\}$ of $\mathfrak{g}(\mu_1,\mu_2)$:

(1.4)

$$E_0 = \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), E_1 = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), E_2 = \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

Then the commutation relation of $\mathfrak{g}(\mu_1, \mu_2)$ is given by

$$[E_1, E_2] = 0,\ [E_2, E_0] = -\mu_2 E_2,\ [E_0, E_1] = \mu_1 E_1.$$

The left translation of E_0, E_1, E_2 are the vector fields $e_0 = \frac{\partial}{\partial x^0}$, $e_1 = e^{\mu_1 x^0}\frac{\partial}{\partial x^1}$, $e_2 = e^{\mu_2 x^0}\frac{\partial}{\partial x^2}$, respectively such that

$$\begin{aligned} &\langle e_0, e_0\rangle = -1,\ \langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 1, \\ &\langle e_i, e_j\rangle = 0 \text{ if } i \neq j. \end{aligned}$$

That is, $\{e_0, e_1, e_2\}$ forms a Lorentzian frame field on $(\mathbb{R}^3, g_{(\mu_1,\mu_2)})$. Hence we see that $\{E_0, E_1, E_2\}$ forms an orthonormal basis for $\mathfrak{g}(\mu_1, \mu_2)$.

For $X \in \mathfrak{g}(\mu_1, \mu_2)$, denote by $\mathrm{ad}(X)^*$ the *adjoint* operator of $\mathrm{ad}(X)$ i.e. it is defined by the equation

$$\langle \mathrm{ad}(X)(Y), Z\rangle = \langle Y, \mathrm{ad}(X)^*(Z)\rangle$$

for any $Y, Z \in \mathfrak{g}(\mu_1, \mu_2)$. Here $\mathrm{ad}(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$. Let U be the symmetric bilinear operator on $\mathfrak{g}(\mu_1, \mu_2)$ defined by

$$U(X, Y) := \frac{1}{2}\{\mathrm{ad}(X)^*(Y) + \mathrm{ad}(Y)^*(X)\}.$$

Lemma 1.1 *Let $\{E_0, E_1, E_2\}$ be the orthonormal basis for $\mathfrak{g}(\mu_1, \mu_2)$ defined in* (1.4)*. Then*

$$\begin{aligned} &U(E_0, E_0) = 0,\ U(E_1, E_1) = \mu_1 E_0,\ U(E_2, E_2) = \mu_2 E_0, \\ &U(E_0, E_1) = \frac{\mu_1}{2} E_1,\ U(E_1, E_2) = 0,\ U(E_2, E_0) = \frac{\mu_2}{2} E_2. \end{aligned}$$

Let $\mathfrak{D}$ be a simply connected domain and $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ an immersion. φ is said to be *timelike* if the induced metric I on $\mathfrak{D}$ is Lorentzian. The induced Lorentzian metric I determines a Lorentz conformal structure $\mathcal{C}_I$ on $\mathfrak{D}$. Let (t, x) be a Lorentz isothermal coordinate system with respect to the conformal structure $\mathcal{C}_I$. Then the first fundamental form I is written in terms of (t, x) as

$$I = e^{\omega}(-dt^2 + dx^2). \tag{1.5}$$

The conformality condition is given in terms of (t, x) by

$$\begin{aligned} &\left\langle \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x}\right\rangle = 0, \\ -&\left\langle \frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial t}\right\rangle = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial x}\right\rangle = e^{\omega}. \end{aligned} \tag{1.6}$$

A conformal timelike surface is called a *Lorentz surface.* Let $u := t + x$ and $v := -t+x$. Then (u, v) defines a null coordinate system with respect to the conformal structure $\mathcal{C}_I$. The first fundamental form I is written in terms of (u, v) as

$$I = e^{\omega} du dv. \tag{1.7}$$

The partial derivatives $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$ are computed to be

$$\frac{\partial \varphi}{\partial u} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right), \quad \frac{\partial \varphi}{\partial v} = \frac{1}{2}\left(-\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\right). \tag{1.8}$$

The conformality condition (1.6) can be written in terms of null coordinates as

$$\begin{aligned} \left\langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial u} \right\rangle &= \left\langle \frac{\partial \varphi}{\partial v}, \frac{\partial \varphi}{\partial v} \right\rangle = 0, \\ \left\langle \frac{\partial \varphi}{\partial u}, \frac{\partial \varphi}{\partial v} \right\rangle &= \frac{1}{2} e^{\omega}. \end{aligned} \tag{1.9}$$

Definition 1.1 Let $\mathfrak{D}(t, x)$ be a simply connected domain. A smooth timelike immersion $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ is said to be *harmonic* if it is a critical point of the energy functional[1]

$$E(\varphi) = \int_{\mathfrak{D}} e(\varphi) dt dx, \tag{1.10}$$

where $e(\varphi)$ is the *energy density* of φ

$$e(\varphi) = \frac{1}{2}\left\{ - \left| \varphi^{-1} \frac{\partial \varphi}{\partial t} \right|^2 + \left| \varphi^{-1} \frac{\partial \varphi}{\partial x} \right|^2 \right\}. \tag{1.11}$$

$\left|\varphi^{-1}\frac{\partial \varphi}{\partial t}\right|^2 = \left\langle \varphi^{-1}\frac{\partial \varphi}{\partial t}, \varphi^{-1}\frac{\partial \varphi}{\partial t} \right\rangle < 0$ and $\left|\varphi^{-1}\frac{\partial \varphi}{\partial x}\right|^2 = \left\langle \varphi^{-1}\frac{\partial \varphi}{\partial x}, \varphi^{-1}\frac{\partial \varphi}{\partial x} \right\rangle > 0$, so $e(\varphi) > 0$ and hence $E(\varphi) \geq 0$.

Lemma 1.2 *Let $\mathfrak{D}$ be a simply connected domain. A smooth timelike immersion $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ is harmonic if and only if it satisfies the wave equation*

$$\begin{aligned} &-\frac{\partial}{\partial t}\left(\varphi^{-1}\frac{\partial \varphi}{\partial t}\right) + \frac{\partial}{\partial x}\left(\varphi^{-1}\frac{\partial \varphi}{\partial x}\right) \\ &\quad - \left\{ -\mathrm{ad}\left(\varphi^{-1}\frac{\partial \varphi}{\partial t}\right)^* \left(\varphi^{-1}\frac{\partial \varphi}{\partial t}\right) + \mathrm{ad}\left(\varphi^{-1}\frac{\partial \varphi}{\partial x}\right)^* \left(\varphi^{-1}\frac{\partial \varphi}{\partial x}\right) \right\} = 0. \end{aligned} \tag{1.12}$$

[1] This is an analogue of the Dirichlet energy.

Proof. Let $\varphi_s,\ s \in (-\epsilon, \epsilon)$ be a smooth variation of $\varphi = \varphi_0$ such that $\varphi_s|_{\partial\mathfrak{D}} = \varphi|_{\partial\mathfrak{D}}$, where $\partial\mathfrak{D}$ is the boundary $\mathfrak{D}$. Let

$$\Lambda = \frac{d}{ds}(\varphi^{-1}\varphi_s)|_{s=0} : \mathfrak{D} \longrightarrow \mathfrak{g}(\mu_1, \mu_2).$$

$$\begin{aligned}
\frac{d}{ds}E(\varphi_s)|_{s=0} &= \int_{\mathfrak{D}} \left\{ -\left\langle \frac{d}{ds}\left(\varphi^{-1}\frac{\partial\varphi}{\partial t}\right)|_{s=0}, \varphi^{-1}\frac{\partial\varphi}{\partial t} \right\rangle \right. \\
&\quad \left. + \left\langle \frac{d}{ds}\left(\varphi^{-1}\frac{\partial\varphi}{\partial x}\right)|_{s=0}, \varphi^{-1}\frac{\partial\varphi}{\partial x} \right\rangle \right\} dtdx \\
&= \int_{\mathfrak{D}} \left\{ -\left\langle \left[\varphi^{-1}\frac{\partial\varphi}{\partial t}, \Lambda\right] + \frac{\partial\Lambda}{\partial t}, \varphi^{-1}\frac{\partial\varphi}{\partial t} \right\rangle \right. \\
&\quad \left. + \left\langle \left[\varphi^{-1}\frac{\partial\varphi}{\partial x}, \Lambda\right] + \frac{\partial\Lambda}{\partial x}, \varphi^{-1}\frac{\partial\varphi}{\partial x} \right\rangle \right\} dtdx \\
&= \int_{\mathfrak{D}} \left\langle \Lambda, -\frac{\partial}{\partial t}\left(\varphi^{-1}\frac{\partial\varphi}{\partial t}\right) + \frac{\partial}{\partial x}\left(\varphi^{-1}\frac{\partial\varphi}{\partial x}\right) \right. \\
&\quad \left. + \operatorname{ad}\left(\varphi^{-1}\frac{\partial\varphi}{\partial t}\right)^{*}\left(\varphi^{-1}\frac{\partial\varphi}{\partial t}\right) - \operatorname{ad}\left(\varphi^{-1}\frac{\partial\varphi}{\partial x}\right)^{*}\left(\varphi^{-1}\frac{\partial\varphi}{\partial x}\right) \right\rangle \\
&\quad dtdx.
\end{aligned}$$

This completes the proof.

In terms of null coordinates u, v, the wave equation (1.12) can be written as

$$\frac{\partial}{\partial u}\left(\varphi^{-1}\frac{\partial\varphi}{\partial v}\right) + \frac{\partial}{\partial v}\left(\varphi^{-1}\frac{\partial\varphi}{\partial u}\right) - 2U\left(\varphi^{-1}\frac{\partial\varphi}{\partial u}, \varphi^{-1}\frac{\partial\varphi}{\partial v}\right) = 0. \tag{1.13}$$

Let $\varphi^{-1}d\varphi = \alpha' du + \alpha'' dv$. Then the equation (1.13) is equivalent to

$$\alpha'_v + \alpha''_u = 2U(\alpha', \alpha''). \tag{1.14}$$

The Maurer-Cartan equation is given by

$$\alpha'_v - \alpha''_u = [\alpha', \alpha'']. \tag{1.15}$$

The equations (1.14) and (1.15) can be combined to a single equation

$$\alpha'_v = U(\alpha', \alpha'') + \frac{1}{2}[\alpha', \alpha'']. \tag{1.16}$$

The equation (1.16) is both the integrability condition for the differential equation $\varphi^{-1}d\varphi = \alpha' du + \alpha'' dv$ and the condition for φ to be a harmonic map.

The Levi-Civita connection ∇ of $G(\mu_1, \mu_2)$ is computed to be

$$\begin{aligned}
&\nabla_{e_0} e_0 = 0,\ \nabla_{e_0} e_1 = -\mu_1 e_1,\ \nabla_{e_0} e_2 = -\mu_2 e_2,\\
&\nabla_{e_1} e_0 = -\mu_1 e_1,\ \nabla_{e_1} e_1 = -\mu_1 e_0,\ \nabla_{e_1} e_2 = 0,\\
&\nabla_{e_2} e_0 = -\mu_2 e_2,\ \nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -\mu_2 e_0.
\end{aligned}$$

Let $K(e_i, e_j)$ denote the sectional curvature of $G(\mu_1, \mu_2)$ with respect to the tangent plane spanned by e_i and e_j for $i, j = 0, 1, 2$. Then

$$\begin{aligned}
K(e_0, e_1) &= g^{00} R^1_{010} = \mu_1^2,\\
K(e_1, e_2) &= g^{11} R^2_{121} = \mu_1 \mu_2,\\
K(e_0, e_3) &= g^{00} R^3_{030} = \mu_2^2,
\end{aligned} \tag{1.17}$$

where $g_{ij} = g_{(\mu_1,\mu_2)}(e_i, e_j)$ denotes the metric tensor of $G(\mu_1, \mu_2)$. Hence, we see that $G(\mu_1, \mu_2)$ has constant sectional curvature if and only if $\mu_1^2 = \mu_2^2 = \mu_1\mu_2$. If $c := \mu_1 = \mu_2$, then $G(\mu_1, \mu_2)$ is locally isometric to $\mathbb{S}^3_1(c^2)$, the de Sitter 3-space of constant sectional curvature c^2. (See Example 1.2 and Remark 1.1.) If $\mu_1 = -\mu_2$, then $\mu_1 = \mu_2 = 0$, so $G(\mu_1, \mu_2) = G(0, 0)$ is locally isometric to $\mathbb{E}^3_1$ (Example 1.1).

Example 1.1 (Minkowski 3-space) The Lie group $G(0, 0)$ is isomorphic and isometric to the Minkowski 3-space

$$\mathbb{E}^3_1 = (\mathbb{R}^3(x^0, x^1, x^2), +)$$

with the metric $-(dx^0)^2 + (dx^1)^2 + (dx^2)^2$.

Example 1.2 (de Sitter 3-space) Take $\mu_1 = \mu_2 = c \neq 0$. Then $G(c, c)$ is the flat chart model of the de Sitter 3-space:

$$\mathbb{S}^3_1(c^2)_+ = (\mathbb{R}^3(x^0, x^1, x^2), -(dx^0)^2 + e^{-2cx^0}\{(dx^1)^2 + (dx^2)^2\}).$$

Remark 1.1 Let $\mathbb{E}^4_1$ be the Minkowski 4-space. The natural Lorentzian metric $\langle\cdot,\cdot\rangle$ of $\mathbb{E}^4_1$ is expressed as

$$\langle\cdot,\cdot\rangle = -(du^0)^2 + (du^1)^2 + (du^2)^2 + (du^3)^2.$$

in terms of natural coordinate system (u^0, u^1, u^2, u^3). The *de Sitter* 3-*space* $\mathbb{S}^3_1(c^2)$ of constant sectional curvature $c^2 > 0$ is realized as the hyperquadric in $\mathbb{E}^4_1$:

$$\mathbb{S}^3_1(c^2) = \{(u^0, u^1, u^2, u^3) \in \mathbb{E}^4_1 :\ -(u^0)^2 + (u^1)^2 + (u^2)^2 + (u^3)^2 = 1/c^2\}.$$

The de Sitter 3-space $\mathbb{S}^3_1(c^2)$ is divided into the following three regions:

$$\mathbb{S}^3_1(c^2)_+ = \{(u^0,u^1,u^2,u^3) \in \mathbb{S}^3_1(c^2) : \ c(u^0+u^1) > 0\};$$
$$\mathbb{S}^3_1(c^2)_0 = \{(u^0,u^1,u^2,u^3) \in \mathbb{S}^3_1(c^2) : \ u^0+u^1 = 0\};$$
$$\mathbb{S}^3_1(c^2)_- = \{(u^0,u^1,u^2,u^3) \in \mathbb{S}^3_1(c^2) : \ c(u^0+u^1) < 0\}.$$

$\mathbb{S}^3_1(c^2)$ is the disjoint union $\mathbb{S}^3_1(c^2)_+ \dotplus \mathbb{S}^3_1(c^2)_0 \dotplus \mathbb{S}^3_1(c^2)_-$ and $\mathbb{S}^3_1(c^2)_\pm$ are diffeomorphic to $\mathbb{R}^3$. Let us introduce a local coordinate system (x^0,x^1,x^2) by

$$x^0 = \frac{1}{c}\log c(u^0+u^1), \quad x^j = \frac{u^{j+1}}{c(u^0+u^1)}, \ (j=1,2).$$

This local coordinate system is defined on $\mathbb{S}^3_1(c^2)_+$. The induced metric of $\mathbb{S}^3_1(c^2)_+$ is expressed as:

$$g_c := -(dx^0)^2 + e^{2cx^0}\{(dx^1)^2+(dx^2)^2\}.$$

The chart $(\mathbb{S}^3_1(c^2)_+, g_c)$ is traditionally called the *flat chart* of $\mathbb{S}^3_1(c^2)$ in general relativity [2]. The flat chart is identified with a Lorentzian manifold

$$\mathbb{R}^3_1(c^2) := (\mathbb{R}^3, -(dx^0)^2 + e^{2cx^0}\{(dx^1)^2+(dx^2)^2\})$$

of constant sectional curvature c^2. This expression shows that the flat chart is a warped product $\mathbb{E}^1_1 \times_f \mathbb{E}^2$ with warping function $f(x^0) = e^{cx^0}$. In particular, $\mathbb{S}^3_1(c^2)_+$ is a Robertson-Walker spacetime.

Example 1.3 (Direct product $\mathbb{E}^1 \times \mathbb{R}^2_1(c^2)$) Take $(\mu_1,\mu_2) = (0,c)$ with $c \neq 0$. Then the resulting homogeneous spacetime is $\mathbb{R}^3$ with metric:

$$-(dx^0)^2 + (dx^1)^2 + e^{-2cx^0}(dx^2)^2,$$

or equivalently,

$$(dx^1)^2 - (dx^0)^2 + e^{-2cx^0}(dx^2)^2,$$

Hence $G(0,c)$ is identified with the direct product of the real line $\mathbb{E}^1(x^1)$ and the warped product model

$$\mathbb{R}^2_1(c^2) = (\mathbb{R}^2(x^0,x^2), -(dx^0)^2 + e^{-2cx^0}(dx^2)^2)$$

of $\mathbb{S}^2_1(c^2)_+$. Here, $\mathbb{R}^2_1(c^2)$ denotes the flat chart model of $\mathbb{S}^2_1(c^2)$. Thus $G(0,c)$ is identified with $\mathbb{E}^1 \times \mathbb{R}^2_1(c^2)$. Note that $G(0,c)$ is a warped product with trivial warping function.

Example 1.4 (Homogeneous spacetime $G(c,-c)$) Let $\mu_1 = c, \mu_2 = -c$ with $c \neq 0$. Then the resulting homogeneous spacetime $G(c,-c)$ is the Minkowski motion group $E(1,1)$ with the Lorentzian metric:

$$-(dx^0)^2 + e^{-2cx^0}(dx^1)^2 + e^{2cx^0}(dx^2)^2.$$

2 Integral representation formula

Let $\mathfrak{D}(u,v)$ be a simply connected domain and $\varphi : \mathfrak{D} \longrightarrow G(\mu_1,\mu_2)$ an immersion. Let us write $\varphi(u,v) = (x^0(u,v), x^1(u,v), x^2(u,v))$. Then

$$(2.1)\qquad \begin{aligned} \alpha' &= \varphi^{-1}\frac{\partial \varphi}{\partial u} \\ &= \frac{\partial x^0}{\partial u}E_0 + \frac{\partial x^1}{\partial u}e^{-\mu_1 x^0}E_1 + \frac{\partial x^2}{\partial u}e^{-\mu_2 x^0}E_2 \end{aligned}$$

and

$$(2.2)\qquad \begin{aligned} \alpha'' &= \varphi^{-1}\frac{\partial \varphi}{\partial v} \\ &= \frac{\partial x^0}{\partial v}E_0 + \frac{\partial x^1}{\partial v}e^{-\mu_1 x^0}E_1 + \frac{\partial x^2}{\partial v}e^{-\mu_2 x^0}E_2. \end{aligned}$$

It follows from (1.14) that

Lemma 2.1 *φ is harmonic if and only if it satisfies the following equations:*

$$(2.3)\qquad \begin{aligned} \frac{\partial^2 x^0}{\partial u \partial v} - \left(\mu_1 \frac{\partial x^1}{\partial u}\frac{\partial x^1}{\partial v}e^{-2\mu_1 x^0} + \mu_2 \frac{\partial x^2}{\partial u}\frac{\partial x^2}{\partial v}e^{-2\mu_2 x^0}\right) &= 0, \\ \frac{\partial^2 x^1}{\partial u \partial v} - \mu_1\left(\frac{\partial x^0}{\partial u}\frac{\partial x^1}{\partial v} + \frac{\partial x^0}{\partial v}\frac{\partial x^1}{\partial u}\right) &= 0, \\ \frac{\partial^2 x^2}{\partial u \partial v} - \mu_2\left(\frac{\partial x^0}{\partial u}\frac{\partial x^2}{\partial v} + \frac{\partial x^0}{\partial v}\frac{\partial x^2}{\partial u}\right) &= 0. \end{aligned}$$

The exterior derivative d is decomposed as

$$d = \partial' + \partial'',$$

where $\partial' = \frac{\partial}{\partial u}du$ and $\partial'' = \frac{\partial}{\partial v}dv$ with respect to the conformal structure of $\mathfrak{D}$. Let

$$\begin{aligned} (\omega^0)' &= \frac{\partial x^0}{\partial u}du = \partial' x^0, \\ (\omega^0)'' &= \frac{\partial x^0}{\partial v}dv = \partial'' x^0, \\ (\omega^1)' &= e^{-\mu_1 x^0}\partial' x^1,\ (\omega^2)' = e^{-\mu_2 x^0}\partial' x^2, \\ (\omega^1)'' &= e^{-\mu_1 x^0}\partial'' x^1,\ (\omega^2)'' = e^{-\mu_2 x^0}\partial'' x^2. \end{aligned}$$

Then by Lemma 2.1, the 1-forms $(\omega_i)'$, $(\omega_i)''$, $i = 0, 1, 2$ satisfy the differential system:

$$\partial''(\omega^0)' = \mu_1(\omega^1)'' \wedge (\omega^1)' + \mu_2(\omega^2)'' \wedge (\omega^2)', \tag{2.4}$$

$$\partial''(\omega^i)' = \mu_i(\omega^i)'' \wedge (\omega^0)', \ i = 1, 2, \tag{2.5}$$

$$\partial'(\omega^0)'' = \mu_1(\omega^1)' \wedge (\omega^1)'' + \mu_2(\omega^2)' \wedge (\omega^2)'', \tag{2.6}$$

$$\partial'(\omega^i)'' = \mu_i(\omega^i)' \wedge (\omega^0)'', \ i = 1, 2. \tag{2.7}$$

Proposition 2.1 *If $(\omega_i)'$, $(\omega_i)''$, $i = 0, 1, 2$ satisfy* (2.4)-(2.7) *on a simply connected domain $\mathfrak{D}$. Then*

$$\varphi(u, v) = \int ((\omega^0)', e^{\mu_1 x^0}(\omega^1)', e^{\mu_2 x^0}(\omega^2)') + \int ((\omega^0)'', e^{\mu_1 x^0}(\omega^1)'', e^{\mu_2 x^0}(\omega^2)'') \tag{2.8}$$

is a harmonic map into $G(\mu_1, \mu_2)$.

Conversely, if $\{(\omega_i)', (\omega_i)'' : i = 0, 1, 2\}$ is a solution to (2.4)-(2.7) and

$$\begin{aligned} -(\omega^0)' \otimes (\omega^0)' + (\omega^1)' \otimes (\omega^1)' + (\omega^2)' \otimes (\omega^2)' = 0, \\ -(\omega^0)'' \otimes (\omega^0)'' + (\omega^1)'' \otimes (\omega^1)'' + (\omega^2)'' \otimes (\omega^2)'' = 0 \end{aligned} \tag{2.9}$$

on a simply connected domain $\mathfrak{D}$, then $\varphi(u, v)$ in (2.8) is a weakly conformal harmonic map into $G(\mu_1, \mu_2)$. In addition, if

$$-(\omega^0)' \otimes (\omega^0)'' + (\omega^1)' \otimes (\omega^1)'' + (\omega^2)' \otimes (\omega^2)'' \neq 0, \tag{2.10}$$

then $\varphi(u, v)$ in (2.8) is a minimal timelike surface in $G(\mu_1, \mu_2)$.

3 Normal Gauß map

Let $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ be a Lorentz surface i.e. a conformal timelike surface. Take a unit normal vector field N along φ. Then by the left translation we obtain the smooth map

$$\varphi^{-1}N : \mathfrak{D} \longrightarrow \mathbb{S}^2_1(1),$$

where

$$\mathbb{S}^2_1(1) = \{u^0E_0 + u^1E_1 + u^2E_2 : -(u^0)^2 + (u^1)^2 + (u^2)^2 = 1\} \subset \mathfrak{g}(\mu_1, \mu_2)$$

is the de Sitter 2-space of constant Gaußian curvature 1. The Lie algebra $\mathfrak{g}(\mu_1, \mu_2)$ is identified with Minkowski 3-space $\mathbb{E}^3_1(u^0, u^1, u^2)$ via the orthonormal basis $\{E_0, E_1, E_2\}$. Then smooth map $\varphi^{-1}N$ is called the normal Gauß map of φ. Let $\varphi : \mathfrak{D} \longrightarrow G(\mu_1, \mu_2)$ be a minimal timelike surface determined by the data $((\omega^0)', (\omega^1)', (\omega^2)')$ and $((\omega^0)'', (\omega^1)'', (\omega^2)'')$. Write $(\omega^i)' = \xi^i du$ and $(\omega^i)'' = \eta^i dv$, $i = 0, 1, 2$. Then

$$
\begin{aligned}
I &= 2(-(\omega^0)' \otimes (\omega^0)'' + (\omega^1)' \otimes (\omega^1)'' + (\omega^2)' \otimes (\omega^2)'') \\
&= 2(-\xi^0\eta^0 + \xi^1\eta^1 + \xi^2\eta^2)dudv.
\end{aligned} \tag{3.1}
$$

The conformality condition (2.9) can be written as

$$
\begin{aligned}
-(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 = 0, \\
-(\eta^0)^2 + (\eta^1)^2 + (\eta^2)^2 = 0.
\end{aligned} \tag{3.2}
$$

It follows from (3.2) that one can introduce pairs of functions (q, f) and (r, g) such that

$$
\begin{aligned}
q = \frac{-\xi^2}{\xi^0 - \xi^1},\ f = \xi^0 - \xi^1, \\
r = \frac{\eta^2}{\eta^0 + \eta^1},\ g = -(\eta^0 + \eta^1).
\end{aligned} \tag{3.3}
$$

In terms of (q, f) and (r, g), $\varphi(u, v) = (x^0(u, v), x^1(u, v), x^2(u, v))$ is given by Weierstraß type representation formula

$$
\begin{aligned}
x^0(u, v) &= \frac{1}{2}\int (1 + q^2)fdu - (1 + r^2)gdv, \\
x^1(u, v) &= -\frac{1}{2}e^{\mu_1 x^0(u,v)}\int (1 - q^2)fdu + (1 - r^2)gdv, \\
x^2(u, v) &= -e^{\mu_2 x^0(u,v)}\int qfdu + rgdv.
\end{aligned} \tag{3.4}
$$

with first fundamental form

$$
I = (1 + qr)^2 fgdudv. \tag{3.5}
$$

Remark 3.1 In the study of minimal timelike surfaces in Minkowski 3-space, one may assume that $f = g = 1$ so that (3.4) reduces to a simpler form called the *normalized Weierstraß formula*. This is possible as there are no restrictions on f and g other than f and g being Lorentz holomorphic and Lorentz anti-holomorphic respectively. (See [3] and [6].) However, this is not the case with minimal timelike surfaces in de Sitter 3-space as we will see later.

It turns out that the pair (q,r) is the Normal Gauß map $\varphi^{-1}N$ projected into the Minkowski 2-pane $\mathbb{E}^2_1$. To see this, first the normal Gauß map is computed to be

$$\varphi^{-1}N = \frac{1}{qr+1}[(q-r)E_0 + (q+r)E_1 + (qr-1)E_2]. \tag{3.6}$$

Let $\wp_{\mathcal{N}} : \mathbb{S}^2_1(1) \setminus \{x^2 = 1\} \longrightarrow \mathbb{E}^2_1 \setminus \mathbb{H}^1_0$ be the stereographic projection from the north pole $\mathcal{N} = (0,0,1)$. Here, $\mathbb{H}^1_0$ is the hyperbola

$$\mathbb{H}^1_0 = \{x^0E_0 + x^1E_1 \in \mathbb{E}^2_1 : -(x^0)^2 + (x^1)^2 = -1\}.$$

Then

$$\wp_{\mathcal{N}}(x^0E_0 + x^1E_1 + x^2E_2) = \frac{x^0}{1-x^2}E^0 + \frac{x^1}{1-x^2}E^1. \tag{3.7}$$

So, the normal Gauß map $\varphi^{-1}N$ is projected into the Minkowski plane $\mathbb{E}^2_1$ via $\wp_{\mathcal{N}}$ as

$$\wp_{\mathcal{N}} \circ \varphi^{-1}N = \frac{q-r}{2}E_0 + \frac{q+r}{2}E_1 \in \mathbb{E}^2_1(t,x). \tag{3.8}$$

In terms of null coordinates (u,v), (3.8) is written as

$$\wp_{\mathcal{N}} \circ \varphi^{-1}N = (q,r) \in \mathbb{E}^2_1(u,v). \tag{3.9}$$

The pair (q,r) is called the *projected normal Gauß map* of φ. It follows from (2.4) and (2.5) that

$$\begin{aligned} \frac{\partial \xi^0}{\partial v} &= \mu_1\eta^1\xi^1 + \mu_2\eta^2\xi^2, \\ \frac{\partial \xi^i}{\partial v} &= \mu_i\eta^i\xi^0,\ \ i = 1,2. \end{aligned} \tag{3.10}$$

Using (3.10), we obtain

$$\begin{aligned} \frac{\partial f}{\partial v} &= \frac{\partial \xi^0}{\partial v} - \frac{\partial \xi^1}{\partial v} \\ &= \frac{\mu_1}{2}(1-r^2)fg + \mu_2 qrfg \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \frac{\partial q}{\partial v} &= -\frac{\frac{\partial \xi^2}{\partial v}f - \xi^2\frac{\partial f}{\partial v}}{f^2} \\ &= -\frac{\mu_1}{2}q(1-r^2)g + \frac{\mu_2}{2}(1-q^2)rg. \end{aligned} \tag{3.12}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned} \frac{\partial \eta^0}{\partial u} &= \mu_1 \xi^1 \eta^1 + \mu_2 \xi^2 \eta^2, \\ \frac{\partial \eta^i}{\partial u} &= \mu_i \xi^i \eta^0, \ \ i = 1, 2. \end{aligned} \tag{3.13}$$

Using (3.13), we obtain

$$\begin{aligned} \frac{\partial g}{\partial u} &= -\frac{\partial \eta^0}{\partial u} - \frac{\partial \eta_1}{\partial u} \\ &= -\frac{\mu_1}{2}(1-q^2)fg - \mu_2 qrfg \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \frac{\partial r}{\partial u} &= -\frac{\frac{\partial \eta^2}{\partial u} g - \eta^2 \frac{\partial g}{\partial u}}{g^2} \\ &= \frac{\mu_1}{2}(1-q^2)rf - \frac{\mu_2}{2} q(1-r^2)f. \end{aligned} \tag{3.15}$$

Remark 3.2 Setting $f = g = 1$, we obtain from (3.11), (3.12), (3.14), and (3.15)

$$\mu_2 qr = -\frac{\mu_1}{2}(1-r^2), \tag{3.16}$$

$$\frac{\partial q}{\partial v} = -\frac{\mu_1}{2}(1-r^2)q + \frac{\mu_2}{2}(1-q^2)r \tag{3.17}$$

and

$$\mu_2 qr = -\frac{\mu_1}{2}(1-q^2), \tag{3.18}$$

$$\frac{\partial r}{\partial u} = \frac{\mu_1}{2}(1-q^2)r - \frac{\mu_2}{2} q(1-r^2). \tag{3.19}$$

It follows from (3.16) and (3.18) that $q = \pm r$. Let $\mu_1 = \mu_2 = c \neq 0$. If $q = r$ then $\frac{\partial q}{\partial v} = \frac{\partial r}{\partial u} = 0$. This means that $q = r$ is a constant, say A. By (3.4) φ is computed to be

$$\begin{aligned} \varphi(u, v) = \Big(& \frac{1}{2}(1 + A^2)(u - v), -\frac{1}{2} e^{\frac{1}{2}c(1+A^2)(u-v)}(1 - A^2)(u + v), \\ & -e^{\frac{1}{2}c(1+A^2)(u-v)}(u + v) \Big) \end{aligned}$$

or

$$\varphi(t, x) = ((1 + A^2)t, -e^{c(1+A^2)t}x, -2e^{c(1+A^2)t}Ax).$$

This surface cannot be minimal as it is not conformal. If $q = -r \neq 0$ then from (3.17) and (3.19) we obtain the separable differential equations

$$\frac{1}{q(1-q^2)}\frac{\partial q}{\partial v} = -c, \tag{3.20}$$

$$\frac{1}{r(1-r^2)}\frac{\partial r}{\partial u} = c. \tag{3.21}$$

(3.20) has solution

$$q\sqrt{\frac{1-q}{1+q}} = A(u)e^{-cv}, \tag{3.22}$$

where $A(u) > 0$ is a Lorentz holomorphic function. (3.21) has solution

$$r\sqrt{\frac{1-r}{1+r}} = B(v)e^{cu}, \tag{3.23}$$

where $B(v) > 0$ is a Lorentz anti-holomorphic function. Since $q = -r$, (3.23) can be written as

$$-q\sqrt{\frac{1+q}{1-q}} = B(v)e^{cu}. \tag{3.24}$$

(3.22) and (3.24) yield

$$q^2 = -A(u)B(v)e^{c(u-v)} < 0.$$

This case cannot occur as q is a real-valued function.

As seen in Section 1, $G(0,0) = \mathbb{E}^3_1$ and $G(c,c) = \mathbb{S}^3_1(c^2)_+$ are the only cases of solvable Lie group $G(\mu_1, \mu_2)$ with constant sectional curvature.

Remark 3.3 For $G(0,0) = \mathbb{E}^3_1$,

$$\frac{\partial f}{\partial v} = \frac{\partial q}{\partial v} = 0,$$
$$\frac{\partial g}{\partial u} = \frac{\partial r}{\partial u} = 0.$$

That is, f, q are Lorentz holomorphic and g, r are Lorentz anti-holomorphic. From (3.4), we retrieve the Weierstraß representation formula ([3], [6]) for minimal timelike surface $\varphi(u,v) = (x^0(u,v), x^1(u,v), x^2(u,v))$ in $\mathbb{E}^3_1$ given by

$$\begin{aligned} x^0(u,v) &= \frac{1}{2}\int (1+q^2)f du - (1+r^2)g dv, \\ x^1(u,v) &= -\frac{1}{2}\int (1-q^2)f du + (1-r^2)g dv, \\ x^2(u,v) &= -\int qf du + rg dv. \end{aligned} \tag{3.25}$$

Remark 3.4 If $\mu_1 = \mu_2 = c \neq 0$, then (3.12) and (3.15) can be written respectively as

$$\frac{\partial q}{\partial v} = \frac{c}{2} g(r-q)(1+qr), \tag{3.26}$$

$$\frac{\partial r}{\partial u} = \frac{c}{2} f(r-q)(1+qr). \tag{3.27}$$

If q is Lorentz holomorphic, then $q = r$ or $1 + qr = 0$. If $1 + qr = 0$ then $I = 0$. $q = r$ cannot occur as discussed in Remark 3.2. Hence, q cannot be Lorentz holomorphic for minimal timelike surfaces in $\mathbb{S}^3_1(c^2)_+$. For the same reason, r cannot be Lorentz anti-holomorphic for minimal timelike surfaces in $\mathbb{S}^3_1(c^2)_+$.

From here on, we assume that $q^2 \neq 1$ and $r^2 \neq 1$. It follows from (3.11), (3.12), (3.14), and (3.15) that the projected normal Gauß map (q, r) satisfies the equations

$$\begin{aligned} &\frac{\partial^2 q}{\partial u \partial v} + \frac{\mu_1(1-r^2) + 2\mu_2 qr}{-\mu_1 q(1-r^2) + \mu_2(1-q^2)r} \frac{\partial q}{\partial u}\frac{\partial q}{\partial v} \\ &+ \frac{(\mu_1^2 - \mu_2^2)(1-q^2)(1+r^2)q}{[-\mu_1 q(1-r^2) + \mu_2(1-q^2)r][-\mu_1(1-q^2)r + \mu_2 q(1-r^2)]} \frac{\partial r}{\partial u}\frac{\partial q}{\partial v} \\ &= 0 \end{aligned} \tag{3.28}$$

and

$$\begin{aligned} &\frac{\partial^2 r}{\partial v \partial u} + \frac{\mu_1(1-q^2) + 2\mu_2 qr}{-\mu_1(1-q^2)r + \mu_2 q(1-r^2)} \frac{\partial r}{\partial u}\frac{\partial r}{\partial v} \\ &+ \frac{(\mu_1^2 - \mu_2^2)(1+q^2)(1-r^2)r}{[-\mu_1(1-q^2)r + \mu_2 q(1-r^2)][-\mu_1 q(1-r^2) + \mu_2(1-q^2)r]} \frac{\partial r}{\partial u}\frac{\partial q}{\partial v} \\ &= 0. \end{aligned} \tag{3.29}$$

The equations (3.28) and (3.29) are not the harmonic map equations for the projected normal Gauß map (q, r) in general. The following theorem tells under what conditions they become the harmonic map equations for (q, r).

Theorem 3.1 *The projected normal Gauß map (q, r) is a harmonic map if and only if $\mu_1^2 = \mu_2^2$. If $\mu_1 = \mu_2 \neq 0$ then (3.28) and (3.29) reduce to*

$$\frac{\partial^2 q}{\partial u \partial v} + \frac{1 - r^2 + 2qr}{(1-q^2)r - q(1-r^2)} \frac{\partial q}{\partial u} \frac{\partial q}{\partial v} = 0, \tag{3.30}$$

$$\frac{\partial^2 r}{\partial v \partial u} + \frac{-(1-q^2) - 2qr}{(1-q^2)r - q(1-r^2)} \frac{\partial r}{\partial u} \frac{\partial r}{\partial v} = 0. \tag{3.31}$$

(3.30) and (3.31) are the harmonic map equations for the map $(q, r) : \mathfrak{D}(u, v) \longrightarrow \left(\mathbb{E}_1^2(\alpha, \beta), \frac{2d\alpha d\beta}{(1-\alpha^2)\beta - \alpha(1-\beta^2)}\right)$. If $\mu_1 = -\mu_2$ then (3.28) and (3.29) reduce to

$$\frac{\partial^2 q}{\partial u \partial v} + \frac{-(1-r^2) + 2qr}{(1-q^2)r + q(1-r^2)} \frac{\partial q}{\partial u} \frac{\partial q}{\partial v} = 0, \tag{3.32}$$

$$\frac{\partial^2 r}{\partial v \partial u} + \frac{-(1-q^2) + 2qr}{(1-q^2)r + q(1-r^2)} \frac{\partial r}{\partial u} \frac{\partial r}{\partial v} = 0. \tag{3.33}$$

(3.32) and (3.33) are the harmonic map equations for the map $(q, r) : \mathfrak{D}(u, v) \longrightarrow \left(\mathbb{E}_1^2(\alpha, \beta), \frac{2d\alpha d\beta}{(1-\alpha^2)\beta + \alpha(1-\beta^2)}\right)$.

Proof. The tension field $\tau(q, r)$ of (q, r) is given by ([1], [9])

$$\tau(q, r) = 4\lambda^{-2} \left(\frac{\partial^2 q}{\partial u \partial v} + \Gamma^{\alpha}_{\alpha\alpha} \frac{\partial q}{\partial u} \frac{\partial q}{\partial v}, \frac{\partial^2 r}{\partial v \partial u} + \Gamma^{\beta}_{\beta\beta} \frac{\partial r}{\partial u} \frac{\partial r}{\partial v} \right), \tag{3.34}$$

where λ is a parameter of conformality. Here, $\Gamma^{\alpha}_{\alpha\alpha}, \Gamma^{\beta}_{\beta\beta}$ are the Christoffel symbols of $\mathbb{E}_1^2(\alpha, \beta)$. Comparing (3.28), (3.29) and $\tau(q, r) = 0$, we see that (3.28) and (3.29) are the harmonic map equations for (q, r) if and only if $\mu_1^2 = \mu_2^2$. In order to find a metric on $\mathbb{E}_1^2(\alpha, \beta)$ with which (3.28) and (3.29) are the harmonic map equations, one needs to solve the first-order partial differential equations

$$
\begin{aligned}
\Gamma^{\alpha}_{\alpha\alpha} &= g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial\alpha} \\
&= \begin{cases} \frac{1-\beta^2+2\alpha\beta}{(1-\alpha^2)\beta-\alpha(1-\beta^2)} & \text{if} \quad \mu_1=\mu_2\neq 0, \\ \\ \frac{-(1-\beta^2)+2\alpha\beta}{(1-\alpha^2)\beta+\alpha(1-\beta^2)} & \text{if} \quad \mu_1=-\mu_2, \end{cases} \\
\Gamma^{\beta}_{\beta\beta} &= g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial\beta} \\
&= \begin{cases} \frac{-(1-\alpha^2)-2\alpha\beta}{(1-\alpha^2)\beta-\alpha(1-\beta^2)} & \text{if} \quad \mu_1=\mu_2\neq 0, \\ \\ \frac{-(1-\alpha^2)+2\alpha\beta}{(1-\alpha^2)\beta+\alpha(1-\beta^2)} & \text{if} \quad \mu_1=-\mu_2. \end{cases}
\end{aligned} \tag{3.35}
$$

The solutions are given by

(3.36)

$$
(g_{\alpha\beta}) = \begin{cases} \begin{pmatrix} 0 & \frac{1}{(1-\alpha^2)\beta-\alpha(1-\beta^2)} \\ \frac{1}{(1-\alpha^2)\beta-\alpha(1-\beta^2)} & 0 \end{pmatrix} & \text{if} \quad \mu_1=\mu_2\neq 0, \\ \\ \begin{pmatrix} 0 & \frac{1}{(1-\alpha^2)\beta+\alpha(1-\beta^2)} \\ \frac{1}{(1-\alpha^2)\beta+\alpha(1-\beta^2)} & 0 \end{pmatrix} & \text{if} \quad \mu_1=-\mu_2. \end{cases}
$$

Q.E.D.

Remark 3.5 Clearly, the projected normal Gauß map (q,r) of a minimal timelike surface in $G(0,0)=\mathbb{E}^3_1$ satisfies the wave equation

$$
\Box(q,r)=0, \tag{3.37}
$$

where $\Box$ denotes the d'Alembertian

$$
\Box=\lambda^{-2}\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right)=4\lambda^{-2}\frac{\partial^2}{\partial u\partial v}. \tag{3.38}
$$

Remark 3.6 Theorem 3.1 tells that Minkowski 3-space $G(0,0)=\mathbb{E}^3_1$, de Sitter 3-space $G(c,c)=\mathbb{S}^3_1$, and $G(c,-c)=E(1,1)$ are the only homogeneous 3-dimensional spacetimes among $G(\mu_1,\mu_2)$ in which the projected normal Gauß map of a minimal timelike surface is harmonic.

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SUNGWOOK LEE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN MISSISSIPPI
118 COLLEGE DRIVE, #5045
HATTIESBURG, MS 39406-0001, U.S.A.

E-mail address: sunglee@usm.edu