**Problems for Exam 2**

Please note that one of these problems will appear on Exam 2. The exam is closed book and closed notes. You will have 50 minutes to solve 4 out of the problems presented for regular credit (each problem is worth 25 points). The remaining problems may be solved for bonus credit (1 point each).

1. Suppose $S \subset \mathbb{R}^n$ is compact, $f : S \to \mathbb{R}$ is continuous, and $f(\mathbf{x}) > 0$ for every $\mathbf{x} \in S$. Show that there is a number $c > 0$ such that $f(\mathbf{x}) \geq c$ for every $\mathbf{x} \in S$.

2. Let $f$ be a linear function from $\mathbb{R}$ to $\mathbb{R}$. Show that if $f$ is continuous at $x = 0$, then $f$ is continuous.

3. Given $S \subset \mathbb{R}^n$, $a \in S$ and $f : S \mapsto \mathbb{R}^m$. Prove that the following are equivalent:

   i) $f$ is continuous at $a$;
   ii) for any sequence $\{x_k\}$ in $S$ that converges to $a$, the sequence $\{f(x_k)\}$ converges to $f(a)$.

4. Show using an $\varepsilon - \delta$ proof that $f(x) = x^2$ is uniformly continuous in $0 < x < 1$.

5. Suppose that a continuously differentiable function $f : \mathbb{R} \mapsto \mathbb{R}$ satisfies $f'(x) = g(f(x)) + h(x)$ for all $x \in \mathbb{R}$, where the functions $g, h : \mathbb{R} \mapsto \mathbb{R}$ are $C^\infty$. Prove that the function $f$ is infinitely differentiable as well.

6. A function $g : S \mapsto \mathbb{R}$ is a convex function if $S$ is a convex set and $\forall x, y \in S$, $0 < \lambda \leq 1$ we have

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y).$$

   Prove that i) a norm is a convex function; and, ii) that if a convex function has a local max (min) at $x$ then it also has a global max (min) at $x$.

   Hint: i) Consider the norm, $||\cdot|| : \mathbb{R}^n \mapsto \mathbb{R}^+ \cup \{0\}$ as a function. It has one property that makes it a convex function. ii) We have a min at $x$ if $g(y) > g(x)$, $\forall x$ in some sufficiently small neighborhood of $S$ around the pont $x$. Examine any other point $z \in S$ and write $y$ as a convex combination of $x$ and $z$. What happens if $\lambda$ is very small?