Title

Spatially-varying Compact Multi-point Flux Approximations for 3-D Adapted Grids with Guaranteed Monotonicity

Abstract

We propose a new single-phase local transmissibility upscaling method for adapted grids in 3-D domains that uses spatially varying and compact multi-point flux approximations (MPFA), based on the VCMP method previously introduced for 2-D Cartesian grids. For each cell face in the coarse upscaled grid, we create a local fine grid region surrounding the face on which we solve three generic local flow problems. The multi-point stencils used to calculate the fluxes across coarse grid cell faces involve up to ten neighboring pressure values. They are required to honor the three generic flow problems as closely as possible while maximizing compactness and ensuring that the flux approximation is as close as possible to being two-point. The resulting MPFA stencil is spatially varying and reduces to two-point expressions in cases without full-tensor anisotropy. Numerical tests show that the method significantly improves upscaling accuracy as compared to commonly used local methods and also compares favorably with a local-global upscaling method. We also present a corrector method that adapts the stencils locally to guarantee that the resulting pressure matrix is an M-matrix. This corrector method is needed primarily for cases where strong permeability contrasts are mis-aligned with the grid locally. The corrector method has negligible impact on the flow accuracy. Finally, we show how the computed MPFA can be used to guide adaptivity of the grid, thus allowing rapid, automatic generation of grids that can account for difficult geologic features such as faults and fractures and efficiently resolve fine-scale features such as narrow channels.

1. Introduction

Subsurface formations may exhibit geometrically very complex geological features that must be well represented in simulations of flow and transport because they can fundamentally impact simulation results. However, to reduce computational costs, simulations are generally performed on grids that are coarse compared to the given geocellular grids. A typical geocellular model of a heterogeneous subsurface formation is composed of 10^7-10^8 cells, and are typically upscaled by a factor of 10 to 1000 in practical reservoir simulation settings. The resulting coarse scale simulation models should retain the key features of the geology and the flow.

Upscaling methods for such simulations need to be designed in such a way that they can accurately represent full-tensor anisotropy. Full-tensor anisotropy is common in subsurface flow models and can be introduced directly by permeability or by grid nonorthogonality effects. It is generally accepted that multi-point flux approximations (MPFA) are required to accurately represent such full-tensor effects in finite-volume based flow models. These methods express the flux between two adjacent grid blocks not only in terms of the pressure in those grid blocks, as in two-point flux approximations (TPFA), but also in terms of pressures in a number of other grid blocks near the face. MPFA methods are more accurate than TPFA methods for systems with full-tensor anisotropy, but come at increased computational costs because of wider discretization stencils. Most MPFA methods introduce a 9-point stencil in 2D and a 27-point stencil in 3D. Also, they often suffer from lack of monotonicity (see Nordbotten, et al. 2006).

In (Lambers, et al. 2008), we proposed the Variable Compact Multi-Point approximation (VCMP), which accommodates full-tensor anisotropy and guarantees a monotone pressure solution. To achieve this we allow the MPFA stencil to vary spatially, we minimize the number of points involved in the stencil, and we let the stencil revert to a TPFA stencil wherever the accuracy obtained with TPFA is sufficiently high. We name our method the Variable Compact Multi-Point method, or VCMP for short. We exploit the added freedom in allowing the stencils to vary from grid cell to grid cell in order to guarantee monotonicity.

VCMP has been implemented in two spatial dimensions on various types of grids with cell-centered finite volume discretizations. In this paper, we describe an extension to 3-D problems, and how VCMP can guide adaptive mesh refinement. We limit ourselves here to single phase upscaling methods, but these transmissibilities can also be used for multi-phase simulations, as shown in (Gerritsen, et al. 2008)

The general design of the VCMP methods is summarized in Section 2. In Section 3, we describe how the monotonicity requirement can be enforced. In Section 4, we discuss integration of upscaling with adaptivity. Section 5 gives results for a test problem. We discuss the results and future directions in Section 6 and list our main conclusions in Section 7.

2. Variable Compact Multi-Point (VCMP) methods

2.1. Fine and coarse grid equations

The upscaling strategy is based on single phase, steady and incompressible flow in a heterogeneous reservoir. The governing dimensionless pressure equation is

$$\nabla \cdot (k \cdot \nabla p) = 0. \qquad (2.1)$$

Here p is the pressure and k the permeability tensor, all of which are non-dimensionalized by appropriate reference values. We ignored any sources or sinks in the domain. This equation is valid on the fine scale, at which we assume that the permeability tensor as given by geostatistical methods is diagonal, constant in each grid cell and may be highly variable in space. It is common practice to keep the same equation as Eqn. (2.1) for the coarse pressure, with the permeability tensor replaced by the coarse scale effective permeability tensor k^* and p by the coarse scale pressure. This has been found to lead to adequate coarse scale pressure solutions (Durlofsky 1991, Pickup et al. 1994, Bourgeat 1984). In the remainder of the paper we will use Eqn. (2.1) indiscriminately of scale.

2.2. The main ideas behind VCMP

We aim to construct a multi-point finite-volume scheme that has five desirable properties:

- The scheme is "close" to a two-point scheme. Reasons are simplicity, efficiency (maximize matrix sparsity) and consistency (the scheme should reduce to a two-point scheme in the case of homogeneous permeability on a Cartesian grid).
- The scheme is applicable to adaptive grid strategies, such as the CCAR strategy developed in Nilsson, et al. (2005). Adaptivity is an effective way to reduce upscaling errors, and improve representation of connected flow paths in highly heterogeneous formations (Gerritsen and Lambers 2008). This is true especially when combined with a multi-point flux approximation.
- The scheme is very accurate for smooth pressure fields. If the pressure field is not smooth, improved accuracy can be achieved by local grid refinement.
- The scheme performs well for grids with a high aspect ratios, such as are commonly used in reservoir simulators.
- The scheme results in an M-matrix. The M-matrix property ensures monotonicity of the pressure solution, and also is desirable for solver efficiency (Stuben 1983).

The M-matrix property is a sufficient condition to obtain a monotone pressure solution, but not a necessary condition. There exist finite-volume schemes with fixed stencils that achieve monotonicity for a large class of problems but satisfy the M-matrix criterion for only a few cases (Aavatsmark 1996). Although this might suggest that requiring an M-matrix is too restrictive to achieve good accuracy for problems with significant full-tensor effects, our experience is that the M-matrix criterion does not limit the accuracy of our finite-volume scheme, when the flux stencil is spatially variable. This observation is consistent with the findings of (Nordbotten 2006).

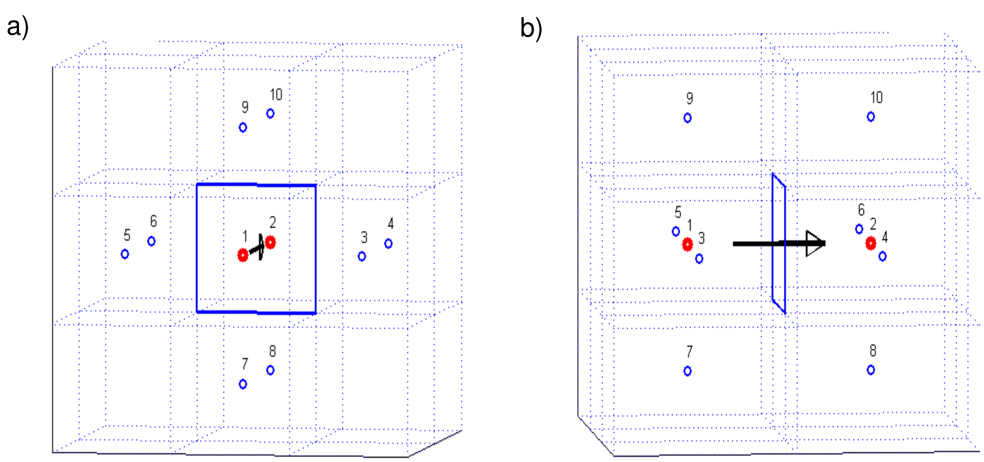

Figure 1: Enumeration of cells is shown for an *x*-oriented face from two different viewpoints: nearly orthogonal to the face in (a), and nearly parallel to the face in (b).

2.3. Construction of VCMP for Cartesian Grids

To create a multi-point flux approximation with the five properties listed above, we allow the MPFA stencil to vary per cell face. Figures 1a and 1b depict an interior face in a Cartesian grid. Our MPFA uses a subset of the ten pressure values p_j, $j = 1, \ldots, 10$ indicated in the figure. For each j, we let t_j denote the weight that will be assigned to point j in the flux approximation, which has the general form

$$f = -\mathbf{t}^T\mathbf{p},$$

where

$$\mathbf{t} = [\, t_1 \ldots t_{10} \,]^T, \quad \mathbf{p} = [\, p_1 \ldots p_{10} \,]^T.$$

We solve the pressure equation on the local region of the fine grid containing the ten points with three generic Dirichlet boundary conditions. We let $\mathbf{p}^1(x,y)$, $\mathbf{p}^2(x,y)$ and $\mathbf{p}^3(x,y)$ be the solutions of these local problems, and p_j^i denote the value of $\mathbf{p}^i(x,y)$ at point j. For both flow problems pressure varies linearly along the boundary of the local region. The pressure field $\mathbf{p}^1$ is computed using boundary values chosen so that the pressure gradient is across the face, and $\mathbf{p}^2$ and $\mathbf{p}^3$ are obtained from boundary values chosen so that the pressure gradient is parallel to the face.

For $i = 1, 2, 3$, we let f_i denote the coarse-scale flux (sum of fine-scale fluxes) across the face obtained from the local solution $\mathbf{p}^i(x,y)$. To compute the weights t_j, we solve the general optimization problem

$$\min_t \sum_{i=1}^{3} \alpha_i^2 \left|\mathbf{t}^T\mathbf{p}^i + f_i\right|^2 + \sum_{j=3}^{10} \beta_j^2 t_j^2 , \qquad (2.2)$$

subject to the essential linear constraints

$$\sum_{j=1}^{10} t_j = 0, \quad t_{2j-1} \le 0, \quad t_{2j} \ge 0, \quad j = 1,\ldots,5. \qquad (2.3)$$

In the current implementation, the weights α_i are chosen to be $|f_i|$ and the weights β_j are chosen to be equal to $(|f_1|+|f_2|+|f_3|)/M$, where M is a tuning parameter. The larger the value of M, the more closely the flows are honored. For small M, we will obtain a two-point flux with $t_j = 0$ for $j = 3,\ldots,10$. In this paper, we use $M = 1000$. We solve this problem using the `lsqlin` function from MATLAB's Optimization Toolbox. If any of the weights are found to be negligibly small, we solve the minimization problem again, with the corresponding variables excluded from consideration. Extension to Cartesian Cell-based Anisotropically Refined (CCAR) grids is discussed in Lambers, et al. (2008).

In the special case of M approaching ∞, we can solve this optimization problem analytically. To accomplish this, we compute up to six 4-point flux stencils that include points 1, 2 and pairs of points chosen from among points 3 through 10. These stencils must honor all 3 flows and satisfy the first condition in equation (2.3), which uniquely determines each stencil. After excluding those stencils which do not satisfy the sign constraints in (2.3), we have a set of N stencils involving points 1, 2, i_k and j_k for $k = 1, \ldots, N$. We then take our MPFA to be a convex combination of these N stencils, with positive weights $w_1, \ldots, w_N$ given by

$$w_k = \frac{\sum_{l,m,n \ne i_k, j_k} t_l^{\,2} t_m^{\,2} t_n^{\,2}}{\sum_{l,m,n} t_l^{\,2} t_m^{\,2} t_n^{\,2}} \qquad (2.4)$$

The indices l, m and n are taken from all points numbered 3 through 10 that occur in any of the stencils. The limitation of six 4-point stencils is due to the fact that these weights are obtained by taking the limit of the solution of a weighted least-squares problem as the weight approaches infinity, to ensure that the weights w_k sum to one, and if more than six stencils are used, then the least-squares problem becomes rank-deficient, and therefore admits infinitely many solutions.

3. The M-fix for Guaranteeing Monotonicity

While VCMP is robust, it does not guarantee an M-matrix. This property is highly desirable, as it ensures monotonicity of the pressure solution, and improves solver efficiency [7]. We therefore employ the M-fix, a corrector method introduced in [8] for the 2-D case. It entails identifying matrix entries with the wrong sign, and recomputing the corresponding MPFAs, with additional constraints chosen in such a way as to guarantee an M-matrix. If the modified optimization problem is infeasible, which rarely occurs in practice, a two-point flux is used instead. As shown in (Chen, et al. 2008), it is not necessary to recompute all fluxes that contribute to wrongly-signed entries, since in practice, not all such entries are likely to cause non-monotonicity in the pressure field for realistic boundary conditions.

4. Application to Adaptive Mesh Refinement

In [1], a grid adaptation strategy was introduced in which cells surrounding a face are refined if, in global coarse-scale flow simulations, a sufficiently large fraction of the total flow passes through the face. However, this causes unnecessary refinement when flow is oriented with the grid. We solve this problem by refining only if the weights t3-t10 are sufficiently large, which only occurs in the presence of full-tensor anisotropy. In addition,
as an alternative to applying the M-fix, we can refine where elements of the matrix have the wrong sign, or where no stencil that satisfies the sign constraints (2) can be computed by VCMP.

5. Numerical Results

We demonstrate our algorithm on a 3-D fine-scale permeability field of size 64 x 64 x 8, which is then upscaled to a 8 x 16 x 8 coarse grid; that is, we do not upscale in the z-direction, since there are very few fine cells along this dimension anyway. The permeability field is illustrated in Figure 2. This field is constructed using portions of layers 9 through 16 of the SPE10 test suite designed by (Christie and Blunt, 2001). One-fourth of each of these layers has been incorporated, rescaled to size 64x64, with fine-scale cells set to 1 unit in each dimension.

In Table 1, we compare the computed total flow from a fine-scale simulation to the flow obtained by solving for the pressure on the coarse grid using (1) a two-point flux, with transmissibilities computed using extended local upscaling, and (2) VCMP, as described in Section 2. For both sets of boundary conditions, we achieve significantly more accuracy using VCMP. It is interesting to note that while TPFA overestimates the total flow, VCMP slightly underestimates it.

Flow from	Fine-scale flow	TPFA	% error	VCMP	% error
x-direction	214	235	9	209	2
y-direction	179	181	1	178	0

Table 1: comparison of total flow obtained from coarse-scale pressure solves using TPFA and VCMP to upscale from the permeability field shown in Figure 2.

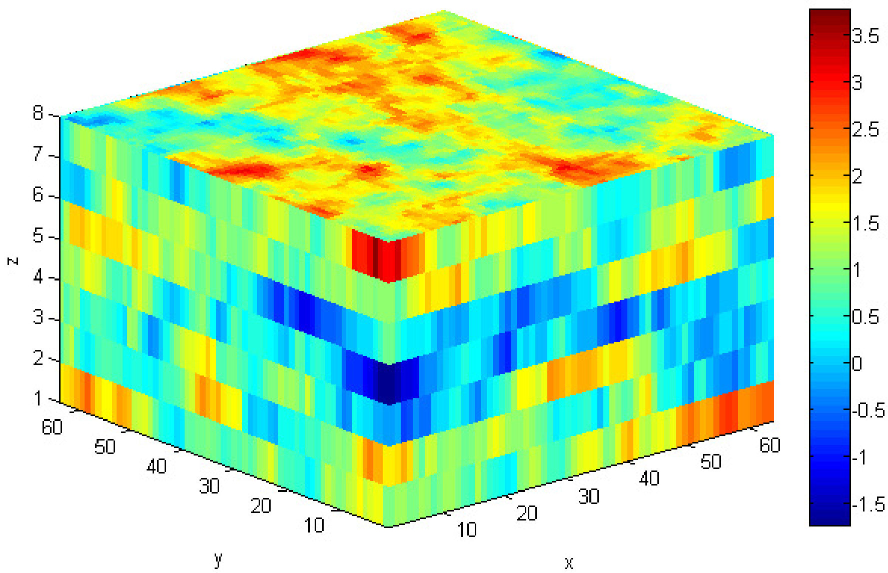

Figure 2: 3-D fine-scale permeability field. The values in the scale to the right are the logarithms, in base 10, of the permeability values.

6. Discussion

The VCMP method described in Section 2 does not strictly guarantee that the pressure matrix will be an M-matrix. Monotonicity problems occur mostly in cases where the permeability is strongly misaligned with the grid and the permeability contrasts are high. However, the M-fix proposed in this paper corrects the matrix and restores monotonicity. Most importantly, it has been shown in Gerritsen, et al. (2006) that the M-fix does not significantly lower the flow accuracy. The M-fix is made possible by the flexibility provided by the VCMP method in choosing the transmissibility coefficients.

It is easy to see that VCMP and the M-fix have straightforward extensions to Cartesian and CCAR grids in three dimensions. In 3D, the VCMP stencil contains no more than 10 points, whereas a full multipoint stencil contains 27 points. The extensions of VCMP to non-orthogonal grids or unstructured grids are possible, as discussed in Lambers, et al. (2008).

7. Summary and Conclusions

We have generalized VCMP to 3-D domains. VCMP accommodates full-tensor anisotropy, which is generally present in coarse-scale flow problems. The stencil adapts to the orientation of the underlying fine permeability distribution, and can be used as an indicator for adaptivity.

Lack of monotonicity may occur in cases where strong permeability contrasts are not aligned with the grid. As in the 2-D case, the M-fix can be used as a corrector step in the VCMP method to guarantee the pressure matrix is an M-matrix.

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