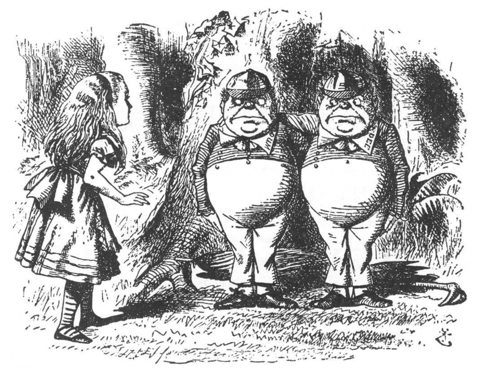


Determinants in Wonderland

The Rev. Charles Lutwidge Dodgson, better known as the author of "Alice in Wonderland," was also a mathematician. He developed an easy, elegant method to compute determinants of matrices. Unfortunately, the method often fails! This poster describes a modification of Dodgson's method that allows it to work for many more matrices—but still not all.

1 Background

Many important problems in science and engineering require us to evaluate the ***determinant*** of a matrix, such as, say

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 3 & 5 & 7 \end{pmatrix}.$$

Algebra students learn to compute determinants by ***expanding cofactors***: [3]

Example 1.

$$\begin{aligned} \det M &= \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 3 & 5 & 7 \end{pmatrix} \\ &= 1 \cdot \det \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 5 \\ 3 & 5 & 7 \end{pmatrix} - 2 \cdot \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 2 & 5 & 7 \end{pmatrix} \\ &\quad + 3 \cdot \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 2 & 3 & 7 \end{pmatrix} - 4 \cdot \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 3 & 5 \end{pmatrix} \\ &= [(3\cdot7-5\cdot5)-(3\cdot7-3\cdot5)+(3\cdot5-3\cdot3)] \\ &\quad -2[(3\cdot7-5\cdot5)-(1\cdot7-2\cdot5)+(1\cdot5-2\cdot3)] \\ &\quad +3[(3\cdot7-3\cdot5)-(1\cdot7-2\cdot5)+(1\cdot3-2\cdot3)] \\ &\quad -4[(3\cdot5-3\cdot3)-(1\cdot5-2\cdot3)+(1\cdot3-2\cdot3)] \\ &= -2. \ \blacklozenge \end{aligned}$$

This is tedious! Dodgson [1] discovered a simpler method to compute the determinant of an $n \times n$ matrix M, where $n \geq 3$:

1. Define the matrix $M_n := M$.
2. Define the matrix M_{n-1} by ***condensing*** M_n:
   take the determinant of every consecutive 2×2 submatrix of M_n.
3. Let $i := n - 2$. Repeat the following until $i = 0$:
   (a) Define the matrix N_i by condensing M_{i+1}.
   (b) Define the matrix M_i by dividing each entry of N_i by the corresponding entry of the interior of M_{i+2}.

Theorem 1 (Dodgson, 1866). *If there is no division by zero, then this method terminates at $i = 1$. The single element of M_1 is $\det M$.* ◆

Example 2. For the matrix M given above, we have:

$$M_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 3 & 5 & 7 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 3 \\ -3 & 6 & -4 \end{pmatrix}$$

$$N_2 = \begin{pmatrix} -4 & 6 \\ -3 & -6 \end{pmatrix} \implies M_2 = \begin{pmatrix} \frac{-4}{2} & \frac{6}{3} \\ \frac{-3}{3} & \frac{-6}{3} \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -1 & -2 \end{pmatrix}$$

$$N_1 = (6) \implies M_1 = \left(\frac{6}{-3}\right) = (-2). \ \blacklozenge$$

This is a lot easier!

Deanna Leggett, John Perry, and Ashley Sanders

Department of Mathematics
University of Southern Mississippi, Hattiesburg, MS 39406
Contact: john.perry@usm.edu

2 The problem

Dodgson's method works only if there are no zeroes in the interior of M, and if there are no zeroes in the interior of any of the M_i's generated by the method.

Example 3. [4] The matrix

$$M = \begin{pmatrix} 2 & -1 & 2 & 1 & -3 \\ 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & -2 & -1 & -1 \\ 2 & 1 & -1 & -2 & -1 \\ 1 & -2 & -1 & -1 & 2 \end{pmatrix}$$

has no zeroes in its interior, but when $i = 3$ we obtain the matrix

$$M_3 = \begin{pmatrix} -15 & 6 & 12 \\ 0 & 0 & 6 \\ 6 & -6 & 8 \end{pmatrix}.$$

The zero in the interior of this matrix makes it impossible to compute M_1. ◆

Can we modify Dodgson's method to work around this?

3 Analysis

Dodgson's method is based on a result of Jacobi [2]:

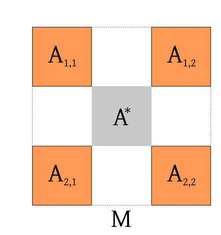
Theorem 2 (Jacobi, 1833). *Let*

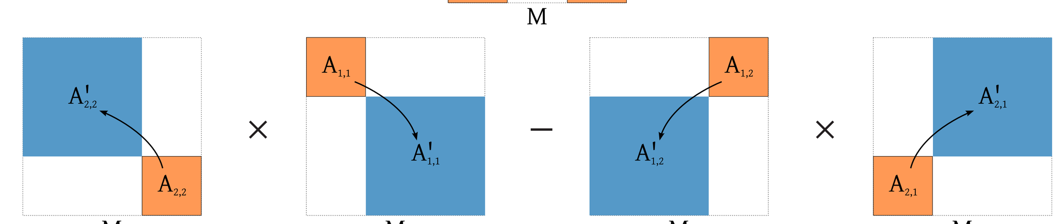
- *M be an $n \times n$ matrix;*
- *A be an $m \times m$ minor of M, where $m < n$;*
- *A' be the corresponding minor of the adjugate of M; and*
- *A^* the complementary $(n-m) \times (n-m)$ minor of M.*

Then

$$\det A' = (\det M)^{m-1} \cdot \det A^*. \ \blacklozenge$$

How does this theorem gives us Dodgson's method?



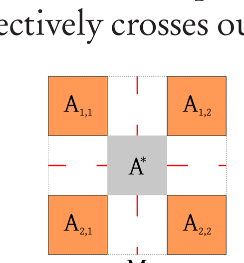


- In the 3×3 matrix M, let the 2×2 minor A consist of the outer corners.
- A' is the corresponding 2×2 minor of the adjugate of M.
  Its elements correspond to the condensations (blue boxes).
- A^* is the interior 1×1 minor of M.
  This is the number by which we eventually divide.

The minor A is 2×2, so $m = 1$, and

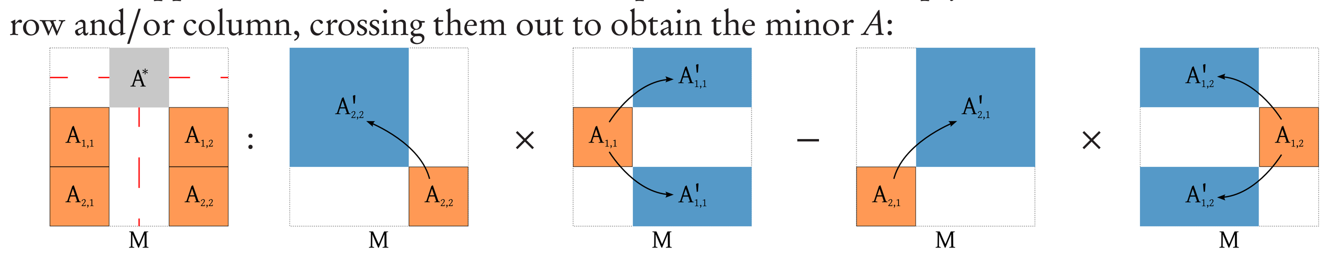
$$\det M = \frac{\det A'}{\det A^*} \implies \det M = \frac{\text{condense twice}}{\text{interior}}.$$

Dodgson's method iterates on 3×3 matrices, "expanding" determinants until they grow to $\det M$. From $i = 2$ on, the method effectively crosses out the row and column containing A^*.



4 A modified Dodgson's method

If a zero appears in the interior, we can adapt the method. Simply choose A^* from a different row and/or column, crossing them out to obtain the minor A:



Example 4. This matrix fails at $i = 2$ using the original Dodgson's method:

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

The modified Dodgson's method works perfectly:

$$M_3 = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 1 & -1 \\ 4 & -2 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad M_1 = (1)$$

Thus $\det M = 1$.

How did we compute M_2? The highlighted zeroes in the interior of M mean that to compute the element in row 1, column 2 of M_2, we have to use a different minor. The elements above the highlighted zeroes are non-zero, so we apply the modification suggested above to find it. To compensate for the zero in row 2, column 3, we crossed out row 1, column 3, obtaining

$$A^* = M_{1,3} \qquad A = \begin{pmatrix} M_{2,2} & M_{2,4} \\ M_{3,2} & M_{3,4} \end{pmatrix} \qquad A' = \begin{pmatrix} \det \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & \det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

We find each elements of A' by crossing out the corresponding element of the 3×3 minor around $M_{2,3}$ and taking the determinant. The careful observer will notice that we already computed $A'_{2,1}$ and $A'_{1,2}$ when computing M_3: $A'_{2,1} = M_{3,(1,2)}$ and $A'_{(1,2)} = M_{3,(1,3)}$. ◆

The method likewise computes the determinant for Example 3, arriving at 36. There are two drawbacks.

1. The modification is not nearly as easy as the original method.
2. If any iteration contains a 3×3 block of zeroes, it fails. An "easy" example:

$$\det \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} = 2^6 = 64.$$

We have succesfully implemented the new method on the computer algebra system Sage. [5]

References

[1] Charles Lutwidge Dodgson, *Condensation of Determinants, being a new and brief method for computing their arithmetical values*, Proceedings of the Royal Society of London **15** (1866), 150–155.

[2] Carl Gustav Jacob Jacobi, *De binis quibuslibet functionibus homogeneis secundi ordinis per substitutiones lineares in alias binas transformandis, quae solis quadratis variabilium constant; una cum variis theorematis de transformatione et determinatione integralium multiplicium*, Journal für die reine und angewandte Mathematik **12** (1833), 1–69.

[3] Pierre-Simon Laplace, *Recherches sur le calcul intégral et sur le système du monde*, Histoire de l'Académie Royale des Sciences (1772), 267–376.

[4] Adrian Rice and Eve Torrence, *"Shutting up like a telescope": Lewis Carroll's "Curious" Condensation Method for Evaluating Determinants*, The College Math Journal **38** (2007), no. 2, 85–95.

[5] William Stein, *Sage: Open Source Mathematical Software (Version 3.1.1)*, The Sage Group, 2008, http://www.sagemath.org.